Introduction to Electrical Engineering Systems

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1 Introduction

1.1 Relevant Units and Terminology

- **Current** is the flow of electrical charge through a circuit element and has units of amperes (A)
- Current, i, can be related to charge, q, by

$$i(t) = \frac{dq(t)}{dt} \therefore q(t) = \int_{t_0}^{t} i(t)\ dt + q\left(t_0\right)$$

- **Direct current** is when a current is constant with time while it is called **alternating current** when it varies periodically with time
- To denote the reference direction, a double-subscript system is used. For a current i_{ab}, the current is from element a to element b
  - For instance, if $i_{ab} = -3$ A, the electrons move from a to b since electrons are negatively charged, and the negative current indicates that the positive charge moves opposite the reference frame
- The **voltage** is the energy transferred per unit of charge that flows through the element and has units of volts (V)
- If a positive charge moves from the positive polarity through toward the negative polarity, the element absorbs energy as heat. If a positive charge moves from the negative polarity toward the positive polarity, energy is supplied
- To denote a reference polarity, a similar double-subscript system is used. For a voltage, v_{ab}, the voltage is between elements a and b with a being the positive reference
- Power, p, can be described by the following and is given in units of watts (W)

$$p = iv$$

- If the current reference enters the positive polarity of the voltage, it is said to have **passive reference configuration**, where a positive power indicates that energy is being absorbed and a negative power indicates the element is supplying energy
- If the current reference enters the negative end of the reference polarity, power should be negated to keep the same terminology in sign
- To calculate the energy, w, delivered to a circuit

$$w = \int_{t_1}^{t_2} p(t)\ dt$$

1.2 Kirchoff's Current and Voltage Laws

- A **node** is a point at which two or more circuit elements are joined together
- Kirchoff's current law states that the net current entering a node is always zero (i.e. input = output)
- All points in a circuit that are connected directly by conductors can be considered to be a single node
- Kirchoff's voltage law states that the sum of the voltages equals zero for any closed loop in an electrical circuit (i.e. input = output)
- When two elements are connected in **series**, the components are connected along a single path such that the same current flows through all of the components

- Two circuit elements are connected in **parallel** if both ends of one element are connected directly to corresponding ends of the other
  - Voltages across parallel elements are equal in magnitude and have the same polarity
    * Therefore, if two voltage sources are shown having different polarities and are in parallel, their voltages (based on this reference polarity) will have opposite signs
- Ohm's law states that resistance, R, is related to current and voltage by

$$v = iR$$

  - If the current reference direction enters the negative reference of the voltage, the equation must be negated
    * Therefore, $v_{ab} = i_{ab}R$ and $v_{ab} = -i_{ba}R$
  - This expression also indicates that $p = Ri^2 = \dfrac{v^2}{R}$
- Resistance can be related to the resistivity, ρ, of a material, a material's length of l, and a material's area of A by

$$R = \frac{\rho L}{A}$$

- There are voltage-controlled voltage sources, current-controlled voltage sources, voltage-controlled current sources, and current-controlled current sources
  - A dependent voltage source is given a diamond symbol with +/- terminals while a dependent current source is given a diamond symbol with an arrow pointing in the direction of the current

2 Resistive Circuits

2.1 Resistances in Series and Parallel

- In this section, the subscripts i and j will represent the resistances of a resistor 1 and resistor 2
- For 2 resistors in series[1],

$$R_{eq} = R_i + R_j$$

- For 2 resistors in parallel[2],

$$R_{eq} = \frac{R_i R_j}{R_i + R_j}$$

- To solve problems, it is frequently easiest to simplify the circuit diagram using equivalent resistances, solve the simplified diagram for any relevant unknowns, and then restore the individual resistances one by one to solve for more unknowns until all parameters of the original are solved for
- For a series combination of 2 resistors, the voltage division law states that[3]

$$v_i = \frac{R_i}{R_i + R_j} v_{\text{total}}$$

- For a parallel combination of 2 resistors, the current division law states that[4]

$$i_i = \frac{R_j}{R_i + R_j} i_{\text{total}}$$

[1] For n resistors in series $R_{eq} = \sum_{i=1}^{n} R_i$

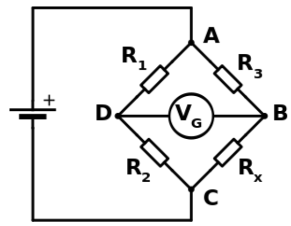
[2] For n resistors in series $R_{eq} = \dfrac{1}{\sum_{i=1}^{n} R_i^{-1}}$

[3] For n resistors in series $v_i = \dfrac{R_i}{\sum_{j=1}^{n} R_j} v_{\text{total}}$

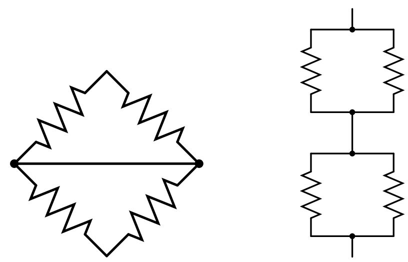
[4] For n resistors in parallel $i_i = \dfrac{R_i^{-1}}{\sum_{j=1}^{n} R_j^{-1}} i_{\text{total}}$

2.2 Circuit Diagram Examples

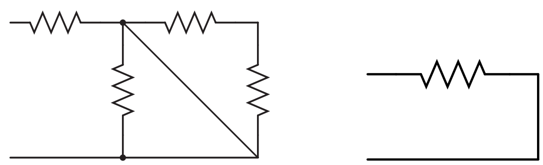
- For the wheatstone bridge shown below, $R_x = \dfrac{R_3}{R_1} R_2$



- The two circuit diagrams below are equivalent:



- The diagonal wire in the first diagram below makes a short circuit, so it is equivalent to the second diagram



2.3 Node-Voltage Analysis

- Node-voltage analysis can be used on circuits that do not have parallel or series resistors
- A reference node is arbitrarily set as 0 V, and a voltage is labeled at each node connecting three or more elements. KCL is then performed at each node
- To find the current flowing out of node n through a resistance towards node k,

$$\frac{v_n - v_k}{R}$$

- If all nodes are used in equations, they are all not independent, so KVL should be used instead of one of the network equations instead
- The negative polarity for each voltage will be at the reference node

The Rosen Guideline:

1. Select a ground node if one is not already provided
   (a) It is easiest to choose a node with the largest number of voltage sources connected to it
2. Write down the following simple relationships, if applicable:

(a) If a node is conected to a voltage source that is connected to a ground, the voltage at the node is equal to the voltage at the voltage source

(b) If a voltage source connects to a node on both sides, then the node voltages are directly related to one another (by the value of the voltage source)

(c) If a voltage source is connected to a node at which a node equation is written and the other terminal of that source is not connected to the ground, then create a supernode around that voltage source

3. For the remaining nodes and all supernodes, write down a nodal analysis voltage equation
4. Solve the system of equations
5. Use Ohm's Law as necessary

2.4 Thévenin and Nortan Equivalent

- The Thévenin equivalent circuit has an independent voltage source in series with a resistance
- The Thévenin source voltage is always equal to the open-circuit voltage of the network. Therefore, the following is true where the subscripts t, oc, and sc, stand for Thévenin, open-circuit, and short-circuit, respectively:

$$R_t = \frac{v_{oc}}{i_{sc}}$$

  - v_{oc} is known as the Thévenin voltage

- To find the Thévenin equivalent, turn the original circuit to an open circuit and obtain the voltage. Then turn the original circuit into a short circuit (connecting a wire between the ends) and find the current. Anything in parallel with a short will have no current going through it. Finally, compute R_t via the equation above
  - When finding R_{th}, only include the resistances that the current goes through when the sources are zeroed. For instance, if there is a short that is created, the current may go through that element and bypass other elements. Start from the + terminal and trace the path(s) to the - terminal
- If the network does not contain a dependent source, you can replace the voltage source with a short and replace any current sources with open circuits. R_t will simply equal R_{eq} of this new circuit. Note that the end terminals are not forming an open circuit here (they can be essentially ignored)
- The Norton equivalent circuit consists of a circuit with an independent current source in parallel with the Thévenin resistance
- i_{sc} is the Norton current. Therefore,

$$v_t = R_t I_n = R_t I_{sc}$$

- A voltage source in series with a resistance can be replaced by a current source in parallel with the resistance (Norton equivalent circuit) with a current given by Ohm's Law
- A current source in parallel with a resistance can be replaced by a voltage source in series with a resistance attached to the positive polarity of the voltage source
- The maximum power can be given by

$$P = \frac{v_t^2}{4R_t}$$

The Rosen Guideline:

1. If there are no dependent sources:

(a) Replace voltage sources with shorts and current sources with open circuits. Find $R_t = R_{eq}$. Try using soure transformations if difficulty occurs

   i. Try to find $v_t = v_{oc}$ by using nodal analysis (and putting a ground at the bottom)

      A. Calculate I_n from R_t and v_t

   ii. If v_{oc} is too hard to find, try to find $I_n = i_{sc}$

      A. Calculate v_t from i_{sc} and R_t

2. If there are dependent sources:

   (a) Find i_{sc} and v_{oc}

      i. Compute R_t from i_{sc} and v_{oc} as well as I_n if necessary

2.5 Principle of Superposition

- For a system composed of resistances, linear dependent soures, and n independent sources

$$r_T = \sum_{i=1}^{n} r_i$$

  - In the above equation, r_i represents a response of independent source i. A response of source i involves zeroing all other sources aside from i and measuring r_i, where r_i can be a current or voltage

3 Inductance and Capacitance

3.1 Capacitance

- Capacitance is energy stored in electric fields
- In an ideal capacitor, the stored charge (q) is proportional to the capacitance (C) and voltage,

$$q = Cv$$

  - Capacitance is measured in farads (F), which has units of coulombs per volt

- Also,

$$i = C\frac{dv}{dt}$$

  - Note that if it is not passive configuration, the equation should be negated

- The following relationship exists between voltage and current through a capacitor

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(t)\ dt + v(t_0)$$

- The energy stored in a capacitor is given by

$$w(t) = \frac{1}{2}Cv^2(t) = \frac{1}{2}v(t)q(t) = \frac{q^2(t)}{2C}$$

- For a series connection of n capacitors, the equivalent conductance can be computed in the same way as n parallel resistors
- For a parallel connection of n capacitors, the equivalent conductance can be computed in the same way as n series resistors

- For a distance d between the plates of a capacitor of area A,

$$C = \frac{\varepsilon A}{d}$$

- The dielectric constant of materials other than vacuums can be found as

$$\epsilon = \epsilon_r \epsilon_0$$

3.2 Inductance

- Inductance is energy storage in magnetic fields
- Voltage through an inductor can be equated to current by a factor of inductance L such that

$$v(t) = L\frac{di}{dt}$$

  - Inductance has units of henries (H), which is equivalent to volt seconds per ampere

- The following relationship exists between current and voltage through an inductor

$$i(t) = \frac{1}{L}\int_{t_0}^{t} v(t)\ dt + i(t_0)$$

- The energy stored in an inductor is given by

$$w(t) = \frac{1}{2}Li^2(t)$$

- For a series connection of n inductors, the equivalent inductance can be computed in the same way as n series resistors
- For a parallel connection of inductors, the equivalent inductance can be computed in the same way as n parallel resistors

4 Steady-State Sinusoidal Analysis

4.1 Complex Numbers

- The following relationships are needed to convert between Cartesian and polar

$$|Z|^2 = x^2 + y^2$$
$$\tan\theta = \frac{y}{x}$$
$$x = |Z|\cos\theta$$
$$y = |Z|\sin\theta$$

- To convert from polar to Cartesian, find x and y, and plug it into the equation $Z = x + yj$
  - For instance, $Z = 5\angle 30^\circ$ has $x = 5\cos 30^\circ = 4.\bar{3}$, and $y = 5\sin 30^\circ = 2.5$, so $Z = 4.\bar{3} + 2.5j$
- To convert from Cartesian to polar, find $|Z|$ and then find θ
  - For instance, $Z = 10 + 5j$ has $|Z| = \sqrt{10^2 + 5^2} \approx 11.18$ and $\theta = \tan^{-1}\left(\frac{5}{10}\right) \approx 26.57^\circ$, so $Z = 11.18\angle 26.57^\circ$
- Euler's identities state that

$$A\angle\theta = Ae^{j\theta} = A\left(\cos\theta + j\sin\theta\right)$$

4.2 Sinusoidal Currents and Voltage

- A sinusoidal voltage is given by

$$v(t) = V_m \cos(\omega t + \theta)$$

  - The variable ω is angular frequency in radians per second, V_m is the peak voltage, and θ is the phase angle

- Frequency is given as the following where T is period

$$f = \frac{1}{T}$$

- Also,

$$\omega = \frac{2\pi}{T} = 2\pi f$$

- Cosine and sine are related by

$$\sin z = \cos(z - 90^\circ)$$

- The root-mean-square (rms) value of the periodic voltage is

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T v^2(t)\,dt}$$

- The average power is thus

$$P_{avg} = \frac{V_{rms}^2}{R}$$

- For periodic current,

$$I_{rms} = \sqrt{\frac{1}{T}\int_0^T i^2(t)\,dt}$$

- The average power is thus

$$P_{avg} = I_{rms}^2 R$$

- For a sinusoid,

$$V_{rms} = \frac{V_m}{\sqrt{2}}$$

- The time to the first positive peak of a sinusoid is

$$t_{max} = -\frac{\theta}{360^\circ}T$$

4.3 Phasors

4.3.1 Definitions

- A sinusoidal voltage of $v(t) = V_m \cos(\omega t + \theta)$ can be converted to phasor format by

$$\mathrm{V} = V_m \angle \theta$$

  - If $v(t)$ is a function of sin, then the θ in the phasor equation should be replaced by $\theta - 90^\circ$

- To add sinusoids, write the phasor for each term in the sum, add the phasors by using complex-number arithmetic, and write the simplified expression for the sum

- Complex vectors are said to rotate counterclockwise, so to find the phase relationship between voltages or currents, draw the phasor diagram. If a vector is further counterclockwise from another phasor, it leads by the difference in the angles of the two phasors. Alternatively, if a vector is further clockwise from another phasor, it lags by the difference in the angles of the two phasors

4.3.2 Example

Consider $v_1(t) = 20\cos(\omega t - 45°)$ and $v_2(t) = 10\sin(\omega t + 60°)$. Add the sinusoids.

1. First, reduce the sum to a single term
   (a) $\boldsymbol{V}_1 = 20\angle - 45°$ and $\boldsymbol{V}_2 = 10\angle - 30°$
   (b) Therefore, $\boldsymbol{V}_s = 20\angle - 45° + 10\angle - 30°$
2. Convert from polar to Cartesian to add the terms
   (a) $\boldsymbol{V}_s = 14.14 - 14.14j + 8.66 - 5j = 22.80 - 19.14j$
3. Convert from Cartesian back to polar
   (a) $\boldsymbol{V}_s = 29.77\angle - 40.01°$
4. Write the time function of the phasor
   (a) $v_s(t) = 29.77\cos(\omega t - 40.01°)$

4.4 Complex Impedances

- For inductances,

$$Z_L = j\omega L = \omega L\angle 90°$$
$$\boldsymbol{V}_L = Z_L \boldsymbol{I}_L$$

- For capacitors,

$$Z_C = \frac{1}{j\omega C} = \frac{-j}{\omega C} = \frac{1}{\omega C}\angle - 90°$$
$$\boldsymbol{V_C} = Z_C \boldsymbol{I}_C$$

- For resistors,

$$\boldsymbol{V}_R = \boldsymbol{I}_R R$$

- If $Z = \frac{\mathrm{V}}{\boldsymbol{I}}$ is pure real, then the element is a resistance
- If $Z = \frac{\mathrm{V}}{\boldsymbol{I}}$ is pure imaginary and positive, then the element is an inductance
- If $Z = \frac{\mathrm{V}}{\boldsymbol{I}}$ is pure imaginary and negative, then the element is a capacitance
- To algebraically perform an operation of $\frac{V_1\angle\theta_1}{V_2\angle\theta_2}$, simply do $\frac{V_1}{V_2}\angle(\theta_1 - \theta_2)$
  - A value of $\angle 0° = 0j$, $\angle 90° = j$, and $\angle - 90° = -j$

4.5 Circuit Analysis with Phasors and Complex Impedances

4.5.1 KVL and KCL

- The KVL law for phasors is simply the following for a closed loop

$$\sum_i \boldsymbol{V}_i = 0$$

- The KCL law for phasors is simply the following at a node

$$\sum_i \boldsymbol{I} = 0$$

4.5.2 Circuit Analysis with Phasors and Impedances

The method to solve a circuit with phasors is:

1. Replace the time descriptions of the voltage and current sources with phasors. All sources will have the same frequency
2. Replace inductances and capacitances by their complex impedences. Resistances have impedances equal to their resistances
3. Analyze the circuit using methods described earlier
4. Convert back to functions of time

4.6 Thevénin and Norton Equivalent Circuits

- The conventions and methods used here are the same as described for DC sources. Namely,

$$\boldsymbol{V}_t = \boldsymbol{V}_{oc}$$
$$Z_t = \frac{\boldsymbol{V}_{oc}}{\boldsymbol{I}_{sc}}$$
$$\boldsymbol{I}_n = \boldsymbol{I}_{sc}$$

- The Z_t can be found by zeroing all independent sources and finding Z_{eq}
- If the load can take on any complex value, maximum power transfer is attained at

$$Z_{load} = Z_t^*$$

- If the load is required to be a pure resistance, maximum power transfer is attained at

$$Z_{load} = R_{load} = |Z_t|$$

- The current of the load will be

$$\boldsymbol{I}_{load} = \frac{\boldsymbol{V}_t}{Z_t + Z_{load}}$$

- The power of the load is

$$P_{load} = \left| \mathrm{Re}\left(Z_{load}\right) \left(\frac{\boldsymbol{I}_{load}}{\sqrt{2}}\right)^2 \right|$$

  - For maximum power transfer, the impedance of the load should be the complex conjugate of the Thevenin impedance

5 Frequency Response and Resonances

5.1 Fourier Analysis, Filters, and Transfer Functions

- The transfer function is defined as

$$H(f) = \frac{\boldsymbol{V}_{out}}{\boldsymbol{V}_{in}}$$

To find the output voltage from a transfer function plot and an input voltage, the follow method is implemented:

1. Determine the frequency and phasor representation for each input component of the voltage
2. Determine the complex value of the transfer function for each component at the specific frequenecy
3. Multiply each input voltage phasor component by its corresponding transfer function value
4. Convert the output voltage phasors into sinusoids of various frequencies and add them together

5.2 First-Order Lowpass Filters

- The transfer function for a first-order lowpass filter can be given by

$$H(f) = \frac{1}{1 + j\left(\frac{f}{f_B}\right)}$$

where the following is for a first-order RC lowpass filter

$$f_B = \frac{1}{2\pi RC}$$

and the following is for a first-order RL lowpass filter

$$f_B = \frac{R}{2\pi L}$$

- The magnitude of the transfer function is given by

$$|H(f)| = \frac{1}{\sqrt{1 + \left(\frac{f}{f_B}\right)^2}}$$

and the angle is

$$\angle H(f) = -\arctan\left(\frac{f}{f_B}\right)$$

5.3 Decibels, the Cascade Connection, and Logarithmic Frequency Scales

- To convert a transfer-function magnitude to decibels, the following formula is implemented

$$|H(f)|_{dB} = 20 \log |H(f)|$$

- The transfer function of a cascade connection is

$$H(f) = H_1(f) \times H_2(f)$$

- The transfer function in descibels of a cascade connections is

$$|H(f)|_{dB} = |H_1(f)|_{dB} + |H_2(f)|_{dB}$$

- The number of decades is given as

$$\text{decades} = \log\left(\frac{f_2}{f_1}\right)$$

- The number of octaves is given as

$$\text{octaves} = \frac{\log\left(\frac{f_2}{f_1}\right)}{\log 2}$$

5.4 First-Order Highpass Filters

- The transfer function for a first-order lowpass filter can be given by

$$H(f) = \frac{j\left(\frac{f}{f_B}\right)}{1 + j\left(\frac{f}{f_B}\right)}$$

- The magnitude of the transfer function is given by

$$|H(f)| = \frac{\left(\frac{f}{f_B}\right)}{\sqrt{1 + \left(\frac{f}{f_B}\right)^2}}$$

  and the angle is

$$\angle H(f) = 90^\circ - \arctan\left(\frac{f}{f_B}\right)$$

5.5 Ideal Filters

- An ideal lowpass filter passes components below its cutoff frequency and rejects components higher than the cutoff
- An ideal highpass filter passes components above its cutoff frequency and rejects components below the cutoff
- An ideal bandpass filter passes components that lie between its cutoff frequencies and rejects components outside that range
- An ideal band-reject filter rejects components that lie between its cutoff frequencies and passes components outside that range

5.6 Resonance

- The resonant frequency is when the inductance impedance and capacitance impedance are equal in magnitude but with opposite sign
  - For both series and parallel resonance, the resonant frequency is given by

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

- The quality factor is defined as

$$Q = \frac{f_0}{B} = \frac{\omega_0}{\Delta\omega}$$

  where B is the bandwith (difference between the half-power frequencies), ω_0 is the resonance angular frequency, and $\Delta\omega$ is the difference between the half-power angular frequencies

- A half-power frequency is obtained when the transfer-function magnitude drops from its maximum by a factor of $\frac{1}{\sqrt{2}}$