HEAT AND MASS TRANSFER

Based on CHEM_ENG 422 at Northwestern University

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1 INTRODUCTION TO HEAT TRANSFER AND MASS TRANSFER

1.1 HEAT FLOWS AND HEAT TRANSFER COEFFICIENTS

1.1.1 HEAT FLOW

A typical problem in heat transfer is the following: consider a body "A" that exchanges heat with another body, of infinite medium, "B". This can be broken down into either a steady problem or a transient problem. In the steady case, we have (for example) $T_A =$ constant, $T_B =$ constant, and we must find the total heat flow rate, $\dot{Q}$, between A and B. In the transient case, we have (for example) $T_B =$ constant, $T_A = T_A|_{t=0} = T_{A0}$, and we must find T_A as a function of time, t.

In most cases (exceptions: free convection due to density differences and radiation heat transfer), the heat flow rate is proportional to the temperature difference. In other words, from B to A we have

$$\dot{Q} \propto T_B - T_A.$$

Note this means that heat transfer, unlike fluid mechanics, is often a linear problem. It is convenient to define a total, integral heat transfer coefficient H such that

$$\dot{Q} \equiv H(T_B - T_A).$$

For linear problems, H will be independent of T_A and T_B. Also, H depends on physical properties of the bodies, their shapes, the fluid flow, and so on.

1.1.2 AVERAGE HEAT FLUX DENSITY

It is often very useful to consider an average area-based heat transfer coefficient, $\bar{h}$, and an average heat flux density, $\bar{q}$, instead. In this case, we have

$$\bar{q} \equiv \bar{h}(T_B - T_A),$$

where

$$\bar{h} \equiv \frac{H}{A}.$$

With this definition, we can say that

$$\dot{Q} = \bar{q}A.$$

1.1.3 LOCAL HEAT FLUX DENSITY

In general, the heat flux will differ from point to point. It is therefore more accurate to use a local heat flux density, q, and a local area heat transfer coefficient (LAHTC), h. In this case, we have

$$q \equiv h(T_B - T_A),$$

where

$$h \equiv \frac{dH}{dA}.$$

In other words,

$$H = \int h\, dA$$

and

$$\dot{Q} = \int q \, dA.$$

1.1.4 Dimensions

Note the following dimensions for the variables we have defined thus far:

$$[\dot{Q}] = \frac{[\mathrm{J}]}{[\mathrm{s}]}, \quad [q] = \frac{[\mathrm{J}]}{[\mathrm{m}^2][\mathrm{s}]}, \quad [H] = \frac{[\mathrm{J}]}{[\mathrm{s}][\mathrm{K}]}, \quad [h] = \frac{[\mathrm{J}]}{[\mathrm{m}^2][\mathrm{s}][\mathrm{K}]}.$$

1.2 The Differential Equations of Balance

We now wish to derive the very general differential equations of balance that can be used to describe the balance of an arbitrary scalar field, denoted [...], such that there is a conservation of said scalar (i.e. it cannot be created or destroyed). We must first define a few terms. The density, ρ, of [...] is given by

$$[\rho] = \frac{[\dots]}{[\mathrm{m}^3]}.$$

We will also define a flux density, $\vec{J}$, given by

$$[\vec{J}] = \frac{[\dots]}{[\mathrm{m}^2][s]}.$$

If we define a unit normal $\hat{n}$, we can then say that $\vec{J} \cdot \hat{n}$ is the amount of [...] crossing, per unit a time, a surface element of unit area with unit normal in the direction of $\hat{n}$. We will also define the rate of formation, r, of [...] per unit volume as

$$[r] = \frac{[\dots]}{[\mathrm{m}^3][\mathrm{s}]}.$$

We can then, logically, state that the balance of [...] is given (in words) by

$$\begin{aligned} &(\text{total rate of accumulation of } [\dots] \text{ in volume } V) \\ &\qquad = (\text{total rate of inflow of } [\dots] \text{ crossing the surface } S) \\ &\qquad + (\text{total rate of production of } [\dots] \text{ produced in } V). \end{aligned}$$

This is written mathematically as

$$\iiint \frac{\partial \rho}{\partial t} \, dV = -\oint\!\!\oint \vec{J} \cdot \hat{n} \, dS + \iiint r \, dV.$$

Applying the divergence theorem

$$\iiint \frac{\partial \rho}{\partial t} \, dV = -\oint\!\!\oint \operatorname{div}(\vec{J}) \, dS + \iiint r \, dV.$$

We can combine all the integrands together as

$$\iiint \left(\frac{\partial \rho}{\partial t} + \operatorname{div}(\vec{J}) - r \right) dV = 0.$$

Since this must be true over any volume and for any set of integral bounds, the integrand itself must equal zero such that

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\vec{J}) = r.$$

This is the general equation governing all transport phenomena. In words, it is

$$\frac{\partial}{\partial t}(\text{density}) + \operatorname{div}(\text{flux density}) = (\text{rate of production}).$$

In the case where [...] is mass, ρ is the typical density, $\vec{J} = \rho\vec{u}$ and $r = 0$ (if there is no reaction). Then we arrive at the conservation of mass

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\vec{u}) = 0.$$

For incompressible fluids, ρ is constant such that $\operatorname{div}(\vec{u}) = 0$. If instead [...] is charge, ρ is charge density, $\vec{J}$ is current density, and $r = 0$. This leads to the conservation of charge equation. Additionally, if [...] is solute in a dilute solution, ρ is concentration (often instead denoted c) and $\vec{J} = c\vec{u} + \vec{\jmath}$, where (lowercase) $\vec{\jmath}$ is the diffusive flux.

1.3 Mass Transfer Equation

1.3.1 Deriving the Diffusion Equation

With the previous development, we can say that for diffusion of a solute in dilute solution, we arrive at

$$\frac{\partial c}{\partial t} + \operatorname{div}(c\vec{u} + \vec{\jmath}) = r.$$

This equation can be rearranged to the following form:

$$\frac{\partial c}{\partial t} + \operatorname{div}(c\vec{u}) = -\operatorname{div}(\vec{\jmath}) + r.$$

Using the vector calculus rules in Section 3.3, we can rewrite the above expression as

$$\frac{\partial c}{\partial t} + \vec{u} \cdot \operatorname{grad}(c) + c \operatorname{div}(\vec{u}) = -\operatorname{div}(\vec{\jmath}) + r.$$

So far, this analysis has assumed nothing other than a dilute solution. However, now we shall assume that the fluid is incompressible (this is true for most liquids, as well as gases moving at velocities much less than the speed of sound). This implies that $\operatorname{div}(\vec{u}) = 0$ such that

$$\frac{\partial c}{\partial t} + \vec{u} \cdot \operatorname{grad}(c) = -\operatorname{div}(\vec{\jmath}) + r.$$

We now employ Fick's law, which is an empirical relation given by

$$\vec{\jmath} = -D \operatorname{grad}(c),$$

where D is the diffusivity, which depends on solute, solvent, and temperature. In the case of a constant diffusivity,

$$-\operatorname{div}(\vec{\jmath}) = -\operatorname{div}\big(-D \operatorname{grad}(c)\big) = D \operatorname{div}\big(\operatorname{grad}(c)\big) = D\, \nabla^2(c).$$

Plugging this in,

$$\frac{\partial c}{\partial t} + \vec{u} \cdot \mathrm{grad}(c) = D\, \nabla^2(c) + r.$$

Note again that this only applies for dilute solution, incompressible fluid, and constant diffusivity. If you have no flow at all and no chemical reaction, the equation simplifies greatly to

$$\frac{\partial c}{\partial t} = D\, \nabla^2 c,$$

which is appropriately called the diffusion equation in applied mathematics. For steady (time-independent) situations,

$$\nabla^2 c = 0,$$

which is Laplace's equation.

1.3.2 Divergence of Flux Density

It is also worth noting that if we start with the original mass transfer equation at the beginning of this section and only make the assumption of steady-state conditions, we arrive at

$$\mathrm{div}(\vec{J}) = r,$$

which is the steady diffusion equation with chemical reaction. An analogous equation can be written in heat transfer for the steady heat conduction equation, given by

$$\mathrm{div}(\vec{q}) = \Phi,$$

where Φ is the rate of production of heat (instead of mass). These two equations have particular value since they do not rely on Fick's or Fourier's laws and the assumptions that underlie them. However, they more difficult to solve in practice.

1.3.3 Mass Fraction Equations

We shall now return to the equation of balance of solute without chemical reaction but with flow:

$$\frac{\partial c}{\partial t} + \vec{u} \cdot \mathrm{grad}(c) = D\, \nabla^2 \mathrm{c}.$$

We should note that the left-hand terms are the material derivative of concentration, where the material derivative is defined as

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{u} \cdot \nabla.$$

It gives the rate of change as seen by an "observer" moving with the fluid. We now note that one can define concentration via a mass fraction, φ, which is the amount of solute per unit mass of solution. In mathematical terms,

$$\varphi = \frac{c}{\rho},$$

where ρ is the density of solution and c is the concentration of solute. The diffusive flux density is then rewritten as

$$\vec{J} = \varphi\rho\vec{u} + \vec{\jmath}.$$

such that the equation of balance is

$$\frac{\partial(\varphi\rho)}{\partial t} + \operatorname{div}(\varphi\rho\vec{u} + \vec{\jmath}) = r.$$

This can be rewritten as

$$\varphi\frac{\partial\rho}{\partial t} + \rho\frac{\partial\varphi}{\partial t} + \varphi\operatorname{div}(\rho\vec{u}) + \rho\vec{u}\cdot\operatorname{grad}(\varphi) = -\operatorname{div}(\vec{\jmath}) + r.$$

We now recall from the conservation of mass equation that

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\vec{u}) = 0,$$

such that now

$$\rho\frac{\partial\varphi}{\partial t} + \rho\vec{u}\cdot\operatorname{grad}(\varphi) = -\operatorname{div}(\vec{\jmath}) + r.$$

We then see that the left-hand terms are the material derivative of the mass fraction (time density), like we saw when we used concentration except now we did not need to assume an incompressible fluid. The equation becomes a bit more useful in the following (identical) form

$$\rho\frac{D\varphi}{Dt} + \operatorname{div}(\vec{\jmath}) = r.$$

We can generalize our results now to an arbitrary substance […] present in a fluid flow with density ρ and velocity $\vec{u}$, and defining its specific density as the amount of […] present per unit mass of fluid, we have

$$\begin{aligned}&(\text{fluid density})(\text{rate of gain of [...], per unit mass, in the moving fluid element})\\ &\qquad + \operatorname{div}(\text{diffusive flux density of [...]}) = (\text{rate of production of [...]})\end{aligned}$$

1.4 Heat Transfer Equation

To write an analogous equation for heat transfer, we must incorporate the entropy per unit mass, given by $\hat{S}$. With this definition, we can say that

$$\rho T\frac{D\hat{S}}{Dt} = -\operatorname{div}(\vec{q}) + \Phi.$$

Thermodynamics provides the following condition as well

$$T\,dS = C_P dT - \frac{\beta T}{\rho}dP,$$

where

$$\beta \equiv \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P.$$

Plugging this expression into the original differential equation to remove the explicit dependence on entropy yields

$$\rho C_P \frac{DT}{Dt} - \beta T \frac{DP}{Dt} = -\mathrm{div}(\vec{q}) + \Phi.$$

For an ideal gas, $\beta T = 1$. For a liquid, $\beta T \ll 1$ in most cases. Since pressure changes are generally small for liquids, we can say the following under this assumption

$$\rho C_P \left(\frac{\partial T}{\partial t} + \vec{u} \cdot \nabla T\right) = -\mathrm{div}(\vec{q}) + \Phi.$$

We can then employ the common empirical relationship of Fourier's Law, given by

$$\vec{q} = -k\nabla T$$

to get

$$\rho C_P \left(\frac{\partial T}{\partial t} + \vec{u} \cdot \nabla T\right) = k\nabla^2 T + \Phi$$

for constant k (thermal conductivity). The units on thermal conductivity are

$$[k] = \frac{[\mathrm{J}]}{[\mathrm{s}][\mathrm{m}][\mathrm{K}]}$$

This equation can be rewritten as

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \nabla T = \alpha \nabla^2 T + \frac{\Phi}{\rho C_P},$$

where α is the thermal diffusivity given by

$$\alpha \equiv \frac{k}{\rho C_P} = \frac{[\mathrm{length}]^2}{[\mathrm{time}]}.$$

This is the basic equation for heat transfer in a fluid.

In the case of no flow (e.g. for a solid),

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T + \frac{\Phi}{\rho C_P}.$$

If heat generation is absent and there is no flow,

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T,$$

which is commonly referred to as the heat equation.

In the case of steady problems with $\Phi = 0$, we get

$$\vec{u} \cdot \nabla T = \alpha \nabla^2 T$$

In the case of steady problems with no flow (but $\Phi \neq 0$),

$$\nabla^2 T + \frac{\Phi}{k} = 0$$

In the case of steady problems, no flow, and no heat generation,

$$\nabla^2 T = 0,$$

which is the steady heat conduction equation.

Now, these equations only work when the temperature change as a function of pressure is very small (as with a most liquids and solids). This is not necessarily true for gases. As such, when dealing with an ideal gas the appropriate equation would instead be the following

$$\rho C_P \frac{DT}{Dt} = \frac{DP}{Dt} + \mathrm{div}(k\nabla T) + \Phi,$$

which would need to be solved concurrently with the Navier-Stokes equation, continuity equation, and equation of state to determine $\vec{u}$, P, ρ, and T as a function of $\vec{r}$ and t. However, if velocities of gas flow are much smaller than the speed of sound, pressure variations are quite small, and the heat transfer equations for liquids is often appropriate for gases.

2 Canonical Problems in Heat and Mass Transfer

2.1 Shell Energy Balance

2.1.1 Definition

To solve many problems in heat and mass transfer, the method of shell balances must be employed. Effectively, a shell balance (in the context of heat transfer) states that the rate of heat into an infinitely thin shell must equal the heat out of that shell. This will be demonstrated with an example below.

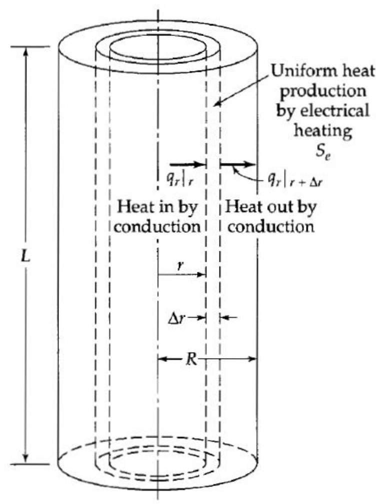
2.1.2 Common Boundary Conditions

There are three very common boundary conditions used in heat transfer. They are as follows.

- The temperature may be specific at a surface
- The heat flux normal to a surface (i.e. the normal component of the temperature gradient) may be given
- At interfaces, the temperature and the heat flux normal to the interface must be continuous
- At a solid-fluid interface, we have that $q = h(T_{\mathrm{H}} - T_{\mathrm{C}})$, where $T_{\mathrm{H}} > T_{\mathrm{C}}$

2.1.3 Heat Conduction through a Cylinder with a Source

Consider the following scenario. A cylindrical wire is internally heated by an electrical current. The heat production per unit volume is given by S_e. The wire has a length L and radius R. The surface of the wire is maintained at a temperature T_0. We can assume that the temperature is only a function of r.



We now consider a cylindrical shell to balance the heat over. There is conduction into the shell at a point r, conduction out of the shell at a point $r + \Delta r$, and energy production by the electricity within that shell. As such, the shell balance should read

$$(Aq_r)|_r - (Aq_r)|_{r+\Delta r} + VS_e = 0$$

The value of A is $2\pi rL$, and the value of V is $2\pi rL\Delta r$. It is best to think about the area as the projection of the cylindrical shell, which is simply the circumference times the length. Similarly, the volume is the aforementioned area times the thickness of the shell, Δr. If we were dealing with spheres, the area term would be $4\pi r^2$, and the volume would be $4\pi r^2 \Delta r$. With this information,

$$(2\pi rLq_r)|_r - (2\pi rLq_r)|_{r+\Delta r} + 2\pi r\Delta rLS_e = 0$$

Dividing through by $2\pi L \Delta r$ and letting $\Delta r \to 0$ yields

$$\frac{d(rq_r)}{dr} = rS_e$$

Integrating this expression yields

$$q_r = \frac{S_e r}{2} + \frac{C_1}{r}$$

The boundary condition is that q_r is finite at $r = 0$, so $C_1 = 0$ and we have

$$q_r = \frac{S_e r}{2}$$

To find the temperature distribution, we use Fourier's law to get

$$-k\frac{dT}{dr} = \frac{S_e r}{2}$$

As such,

$$T = -\frac{S_e r^2}{4k} + C_2$$

The other boundary condition is that $T = T_0$ at $r = R$, so we then get

$$T - T_0 = \frac{S_e R^2}{4k}\left(1 - \left(\frac{r}{R}\right)^2\right)$$

We could have solved this problem using the heat transfer equation (with a source) derived in a prior section to get the same answer.

2.2 The Lumped Capacitance Method

Consider a body that is spatially uniform and which has a spatially uniform temperature distribution (e.g. if it is small and has negligible internal thermal resistance) at a value T_0. Assume that the body is in direct contact with a large heat source (e.g. the atmosphere) at a temperature T_a that does not change temperature. Also, assume that all material properties (e.g. heat capacity), the surface area (A), the volume (V), and $\bar{h}$ (which is not a material property) are known. We wish to describe how the temperature of the body changes with time.

To do so, we first note that the heat flux can be described by

$$\dot{Q} = H(T_a - T)$$

at a given point in time if we wish to describe the temperature flow from the heat source to the body. Since we have the value of $\bar{h}$ and the area we can say that

$$\dot{Q} = \bar{h}A(T_a - T).$$

Recall from thermodynamics that heat capacity, C, is defined as

$$C \equiv \frac{dQ}{dT}.$$

Typically, we wish to refer to a specific heat capacity, which is defined as

$$C_p \equiv \frac{C}{\rho V},$$

Such that

$$dQ = C_p \rho V dT$$

using the previous two equations. We now have an expression for $\dot{Q}$ as well as dQ. We can use the relationship that

$$dQ = \dot{Q} dt$$

to relate the two expressions. As such,

$$C_p \rho V \, dT = \bar{h} A (T_a - T) \, dt.$$

Solving for dT yields

$$dT = \frac{\bar{h} A (T_a - T)}{C_p \rho V} dt.$$

For convenience, we define a timescale τ as

$$\tau \equiv \frac{C}{H} = \frac{C_p \rho V}{\bar{h} A},$$

which has units of time. This makes the differential equation become

$$dT = \frac{T_a - T}{\tau} \, dt.$$

We now define a dimensionless temperature of

$$\theta \equiv \frac{T - T_a}{T_0 - T_a}.$$

This means that at $t = 0$, $\theta = 1$. Further, as the body heats up, $T \to T_a$ and $\theta \to 0$. We now divide the differential equation by $T_0 - T_a$ on both sides to get

$$\frac{dT}{T_0 - T_a} = \frac{T_a - T}{T_0 - T_a} \frac{dt}{\tau}.$$

We note that the left-hand side is equivalent to $d\theta$, and the non-differential terms on the right-right side are $-\theta$. We can then say

$$d\theta = -\frac{\theta}{\tau} dt.$$

Separating the variables to

$$\frac{1}{\theta} d\theta = -\frac{1}{\tau} dt$$

makes the problem easily integrated:

$$\ln(\theta) = -\frac{t}{\tau} + K$$

where K is a constant of integration. If we apply the initial condition of $t = 0$ and $\theta = 1$, we find $K = 0$, and thus

$$\theta = \exp\left(-\frac{t}{\tau}\right).$$

This describes the temperature as a function of time.

2.3 One-Dimensional Slab

One-dimensional problems arise when the geometry and the boundary conditions are such that one can find a coordinate system in which T depends on a single coordinate only. For example, consider a plane (i.e. wall) of "infinite extent" in the y and z directions (commonly referred to as a "slab" geometry) with temperature at each side being fixed. There are two boundary conditions for $T(x)$: $T(0) = T_0$ and $T(L) = T_L$. Assuming a steady solution, no flow, and no heat generation, we arrive at

$$\nabla^2 T = 0,$$

which in this coordinate system is

$$\frac{d^2T}{dx^2} = 0$$

and has the solution

$$T = C_1 + C_2 x.$$

Employing the boundary conditions yields

$$T = T_0 + (T_L - T_0)\frac{x}{L}.$$

Using Fourier's Law allows us to also note that

$$q_x = -k\frac{dT}{dx} = -(T_L - T_0)\frac{k}{L}$$

An analogous problem can be set up for thin film diffusion. Technically, it could not be solved in this way for diffusion through a thick film since a thick film would take a long time to reach diffusive steady state.

2.4 Composite Materials

2.4.1 Concept of Thermal Resistance

From Fourier's Law (with the assumption $\nabla T \approx \Delta T/L$),

$$q = \frac{k}{L}\Delta T \rightarrow \Delta T = \dot{Q}\frac{L}{kA}$$

We can treat this like Ohm's Law where ΔT is analogous to a voltage drop, $\dot{Q}$ is analogous to a current, and L/kA is a resistance. Then, we can define a thermal resistance for conduction as

$$R_{\text{cond}} = \frac{L}{kA}$$

where L is the length in the direction parallel to heat flow and A is the cross-sectional area perpendicular to heat flow.

We can derive a similar expression for convection. The equation for convection is

$$q = \bar{h}\Delta T \rightarrow \Delta T = \dot{Q}\frac{1}{\bar{h}A}$$

Then, we can define a thermal resistance for convection as

$$R_{\text{conv}} = \frac{1}{\bar{h}A}$$

With this information, we can find a total resistance for a composite material by treating these resistances like those of a circuit.

For resistances in series

$$R_{\text{tot}} = \sum_i R_i$$

For resistances in parallel,

$$\frac{1}{R_{\text{tot}}} = \sum_i \frac{1}{R_i}$$

The effective thermal conductivity in a given direction can be found from

$$k_{\text{eff}} = \frac{L}{R_{\text{tot}}A}$$

which can then be used to compute the heat flow rate via

$$\dot{Q} = \frac{\Delta T}{R_{\text{tot}}}$$

2.4.2 THERMAL RESISTANCES IN SERIES

Consider a material consisting of alternating layers in the z direction. This composite material is composed of two different materials, material 1 and material 2, that alternate sequentially and have thicknesses b_1 and b_2. The goal is to find the effective thermal conductivity through this composite material.

Recall that for conduction through the composite,

$$q_z = \frac{k}{L}\Delta T$$

where $\Delta T \equiv T_i - T_f$ and L is a thickness (measured on a path parallel to the heat flow). The net temperature change across the two materials is going to be $\Delta T = \Delta T_1 + \Delta T_2$, the thickness is going to be $L = b_1 + b_2$, and the k is going to be some effective k_{zz} if we consider heat flow in the z direction for now. Therefore,

$$\Delta T = \frac{b_1 + b_2}{k_{zz}} q_z$$

In addition, in each of the two slabs we have

$$\Delta T_1 = \frac{b_1}{k_1} q_z$$

$$\Delta T_2 = \frac{b_2}{k_2} q_z$$

Therefore, using the fact that $\Delta T = \Delta T_1 + \Delta T_2$,

$$\frac{b_1 + b_2}{k_{zz}} q_z = \frac{b_1}{k_1} q_z + \frac{b_2}{k_2} q_z$$

Solving for the effective conductivity

$$k_{zz} = \frac{b_1 + b_2}{\frac{b_1}{k_1} + \frac{b_2}{k_2}}$$

This could have been solved via the thermal resistance method described before. In our example with materials in series,

$$R_{\text{tot}} = \sum_i R_i = R_1 + R_2 = \frac{b_1}{k_1 A} + \frac{b_2}{k_2 A} = \frac{1}{A}\left(\frac{b_1}{k_1} + \frac{b_2}{k_2}\right)$$

This then means that

$$k_{zz} = \frac{b_1 + b_2}{\frac{b_1}{k_1} + \frac{b_2}{k_2}}$$

as we got before.

2.4.3 Thermal Resistances in Parallel

We now consider our prior example but now consider the heat flux in the q_{xx} and q_{yy} directions, such that the materials are parallel to the heat flow. The thermal resistance is as follows (note that in this direction, L is a constant but A is not due to the differences in thickness)

$$\frac{1}{R_{\text{tot}}} = \sum_i \frac{1}{R_i} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{k_1 A_1}{L} + \frac{k_2 A_2}{L} = \frac{W}{L}(k_1 b_1 + k_2 b_2)$$

Therefore,

$$k_{xx} = k_{yy} = \frac{W}{L}(k_1 b_1 + k_2 b_2)\left(\frac{L}{A}\right) = \frac{k_1 b_1 + k_2 b_2}{b_1 + b_2}$$

2.4.4 Thermal Resistances with Convection

We now consider two materials in series that are oriented with heat flow in the z direction. The first material is some standard material with a thickness b_d, and the second material is water that moves through a material with thickness b_f. The water moves very quickly (i.e. is highly turbulent). The two materials alternate sequentially. We can model them as thermal resistors in series, but the first material must use heat conduction whereas the second relies on heat convection. As such,

$$R_{\text{tot}} = \sum_i R_i = \frac{b_d}{k_d A} + \frac{1}{\bar{h}_f A} = \frac{1}{A}\left(\frac{b_d}{k_d} + \frac{1}{\bar{h}_f}\right)$$

Therefore,

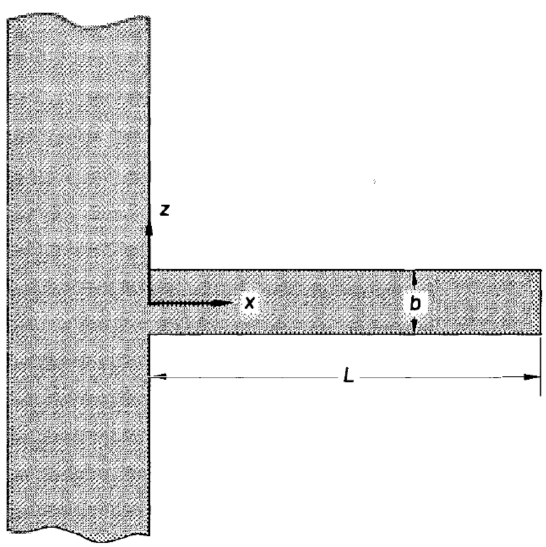
$$k_{zz} = \frac{b_d + b_f}{\dfrac{b_d}{k_d} + \dfrac{1}{\bar{h}_f}}$$

If we assume that $\bar{h}_f \gg b_d/k_d$ due to the highly turbulent nature of the fluid, then the equation simplifies to

$$k_{zz} = \frac{k_d(b_d + b_f)}{b_d}$$

2.5 Rectangular Fin

Consider the fin shown below. Assume that air, at a temperature T_a, surrounds the fin and that the fin is infinitely long in the y direction. Also assume that the left-hand base of the fin in the below image is maintained at a fixed temperature, T_0.



In theory, we now have all the information we need to solve the problem, but the solution will be quite messy. We can make an assumption that we are really only interested in variations in the average temperature, $\langle T\rangle$. Assume that $\langle T\rangle$ is a function of x only. We can then write that the energy balance is

$$\begin{aligned}&(\text{total heat flux in through the cross section in } x)\\ &\quad = (\text{total heat flux out through the cross section at } x + \Delta x)\\ &\quad + (\text{total heat flux out to the fluid through surfaces at } z = \pm\, b/2)\end{aligned}$$

In mathematical form, this is

$$b\langle q_x\rangle|_x = b\langle q_x\rangle|_{x+\Delta x} + 2h(\langle T\rangle - T_\text{a})\Delta x$$

Divide through by Δx and let $\Delta x \to 0$ to yield

$$-b\frac{d\langle q_x\rangle}{dx} = 2h(\langle T\rangle - T_\mathrm{a})$$

We know from Fick's Law that

$$\langle q_x\rangle = -k\frac{d\langle T\rangle}{dx}$$

so

$$bk\frac{d^2\langle T\rangle}{dx^2} = 2h(\langle T\rangle - T_\mathrm{a})$$

This can be rewritten as

$$\frac{d^2\langle T\rangle}{dx^2} - \frac{2h}{bk}(\langle T\rangle - T_\mathrm{a}) = 0$$

We now define the following parameters:

$$\Theta \equiv \frac{\langle T\rangle - T_\mathrm{a}}{T_0 - T_\mathrm{a}}$$

$$\frac{1}{\lambda^2} \equiv \frac{2h}{kb}$$

With these definitions, we can state that

$$\frac{d^2\Theta}{dx^2} = \frac{\Theta}{\lambda^2}$$

The general solution of this equation is the following, which can be determined by assuming a solution of the form $\Theta = Ce^{mx}$ and solving for the characteristics:

$$\Theta = Ae^{-\frac{x}{\lambda}} + Be^{\frac{x}{\lambda}}$$

where A and B are constants of integration. We know that the boundary conditions are $\Theta(x = \infty) = 0$ and $\Theta(x = 0) = 1$. To be clear, even though we did not assume that the fin has a length $L = \infty$, we can still have a boundary condition at $x = \infty$ so long as the length is long enough that the exponential function behaves almost as if you were at $x = \infty$. With these boundary conditions, the solution to the equation is simply

$$\Theta = e^{-x/\lambda}$$

This solution is only approximate due to the aforementioned assumptions. The fin obviously cannot actually be infinitely long in y. We also made the assumption of an $x = \infty$ boundary condition at $x = L$. In addition, we assumed that $T \approx \langle T\rangle$. This last approximation is only valid if

$$\frac{hb}{2k} \ll 1$$

This is commonly referred to as the Biot number, which can be more generally written as the following (where L is a characteristic length, not the L used in the problem statement):

$$\mathrm{Bi} \equiv \frac{hL}{k}$$

Oftentimes, this characteristic length is the volume of the body divided by its surface area.

2.6 Cylindrical Rod

We now consider a cylindrical rod. The left end of the rod is fixed at a temperature T_1. The right end of the rod is fixed at a temperature T_2. For the purposes of this problem, we will set $T_1 > T_2$. In addition, the rod is surrounded by ambient air at a temperature T_a. There is also a heat source given by Φ. We wish to find the temperature distribution in the rod. Assume the rod has a radius R and is oriented along the z axis in cylindrical coordinates. The length of the rod is given by L.

We start with a shell balance. Of course, it will be a cylindrical shell balance that takes into account the heat flux into the rod from the left, out of the rod from the right, out of the rod into the air, and into the rod from the source. It will take the following form.

$$(\pi R^2 q_z)|_z + \pi R^2 \Delta z \Phi = (\pi R^2 q_z)|_{z+\Delta z} + 2\pi R \Delta z h(\langle T \rangle - T_a)$$

where I have assumed that the temperature at $z = \pm r$ is approximately equal to $\langle T \rangle$. Dividing by Δz and letting $\Delta z \to 0$ yields

$$-\frac{dq_z}{dz} + \Phi = \frac{2h(\langle T \rangle - T_a)}{R}$$

Substituting in for Fick's law yields

$$\frac{d^2\langle T \rangle}{dz^2} + \frac{\Phi}{k} = \frac{2h(\langle T \rangle - T_a)}{kR}$$

We have the boundary condition that at $z = 0$, $\langle T \rangle = T_1$ and at $z = L$, $\langle T \rangle = T_2$. If we define the diameter as $d = 2R$ then we can say

$$\frac{d^2\langle T \rangle}{dz^2} - \frac{4h}{kd}\left(\langle T \rangle - \left(T_a + \frac{d}{4h}\Phi\right)\right) = 0$$

This form allows us to get an equation like the one found in the rectangular fin problem. We now introduce the following dimensionless variables:

$$\Theta \equiv \frac{\langle T \rangle - \left(T_a + \frac{d}{4h}\Phi\right)}{T_1 - \left(T_a + \frac{d}{4h}\Phi\right)}$$

$$\frac{1}{\lambda^2} \equiv \frac{4h}{kd}$$

Then we get the equation

$$\frac{d^2\Theta}{dz^2} - \frac{\Theta}{\lambda^2} = 0$$

This equation can once again be integrated to show

$$\Theta = C_1 e^{\frac{z}{\lambda}} + C_2 e^{-\frac{z}{\lambda}}$$

For our purposes, it will be easier to rewrite this with hyperbolic trigonometric functions as

$$\Theta = A\cosh\left(\frac{z}{\lambda}\right) + B\sinh\left(\frac{z}{\lambda}\right)$$

We have the new boundary conditions of at $z = 0$, $\Theta = 1$ and at $z = L$, $\Theta = \frac{\left(T_2 - \left(T_a + \frac{d}{4h}\Phi\right)\right)}{T_1 - \left(T_a + \frac{d}{4h}\Phi\right)} \equiv \Theta_L$. The term $T_a + \frac{d}{4h}\Phi$ has an important meaning. It is the temperature of a long rod with electrical heating that is suspended in the air between non-conducting walls so that it exchanges heat with the air only. This can be proven by simply removing the heat flux terms in the shell balance and solving for $\langle T\rangle$. With the aforementioned boundary conditions, we get

$$\Theta = \cosh\left(\frac{z}{\lambda}\right) + \frac{\Theta_L - \cosh\left(\frac{L}{\lambda}\right)}{\sinh\left(\frac{L}{\lambda}\right)}\sinh\left(\frac{z}{\lambda}\right)$$

If the rod is very long (i.e. $L \gg \lambda$), we expect that the influence of the walls do not extend very far into the rod, so the main part of the rod is really at $T_a + \frac{d}{4h}\Phi$ (i.e. $\Theta = 0$).

2.6.1 Large Internal Resistance

The second case we can consider is when the internal resistance is much greater than the external resistance. In this case,

$$h \gg \frac{k}{L}$$

which is equivalent to saying $\mathrm{Bi} \gg 1$. Upon immersion, the surface temperature instantaneously becomes equal to the temperature of the fluid, say T_1, and remains at this value. Meanwhile, the inside temperature is rising slowly, "propagating" from the boundary to the center. For a large body and slow propagation time, there will be an initial period of time during which the propagation front is still in the "skin layer" far from the center, and so the process can be approximated as transient heat conduction in a semi-infinite slab. The surface will can be treated as flat even if there is curvature to it.

2.7 Semi-Infinite Slab

2.7.1 Mathematical Solution

We now consider transient heat conduction into a semi-infinite slab. Technically, the only requirement is that one dimension is very large or there is very low thermal conductivity such that the distance heat has diffused is small compared to the length of the body. The scenario we consider has said body starting at a temperature T_0 and then jumping up immediately to T_1 on one end. We will only consider a temperature gradient in the x direction of this material. Our initial and boundary conditions are $T = T_0$ at $t = 0, x \leq x < \infty$ and $T = T_1$ at $t > 0, x = 0$. We also know that as $x \to \infty, t \to \infty$ we have $T \to T_0$. We define x here as the distance into the material such that $x = 0$ is at the interface.

We know that

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T$$

which becomes in this problem

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

We can define a dimensionless temperature by

$$\Theta \equiv \frac{T - T_0}{T_1 - T_0}$$

to get a new equation of

$$\frac{\partial \Theta}{\partial t} = \alpha \frac{\partial \Theta^2}{\partial x^2}$$

The new conditions are $\Theta = 0$ at $t = 0, 0 \leq x < \infty$ and $\Theta = 1$ at $t > 0, x = 0$ and the condition of $\Theta \to 0$ as $x \to \infty$ and $t \to \infty$. We use the method of similarity solutions to define a parameter

$$\eta \equiv \frac{x}{\sqrt{4\alpha t}}$$

such that

$$\frac{d^2\Theta}{d\eta^2} + 2\eta \frac{d\Theta}{d\eta} = 0$$

We again have a new set of boundary conditions: $\theta = 0$ at $\eta = \infty$, $\Theta = 1$ at $\eta = 0$, and $\Theta \to 0$ at $\eta = \infty$. The first and third condition are redundant. We can integrate the above equation to get

$$\Theta = 1 - \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\xi^2} d\xi = 1 - \mathrm{erf}(\eta) = \mathrm{erfc}(\eta)$$

where

$$\mathrm{erf}(\eta) \equiv \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\xi^2} d\xi$$

and

$$\mathrm{erfc}(\eta) \equiv 1 - \mathrm{erf}(\eta)$$

We now have an expression for the temperature difference as a function of time. It is important to know that $\mathrm{erf}(0.5) \approx 0.52$ and $\mathrm{erf}(1.8) \approx 0.99$. The maximum value of $\mathrm{erf}(\eta)$ is 1 and the minimum is 0, which makes sense with our definition of Θ. We also note that

$$\frac{d}{d\eta} \mathrm{erf}(\eta) = \frac{2}{\sqrt{\pi}} \frac{d}{d\eta} \int_0^\eta e^{-\xi^2} d\xi = \frac{2}{\sqrt{\pi}} e^{-\eta^2}$$

Therefore,

$$\frac{d}{d\eta}\operatorname{erf}(\eta)\Big|_{\eta=0} = \frac{2}{\sqrt{\pi}}$$

This is useful in finding the heat flux into the medium through the interface. We then perform a bit of algebra (noting that $x = \eta\sqrt{4\alpha t}$)

$$q_x|_{x=0} = -k\frac{\partial T}{\partial x}\Big|_{x=0} = -k(T_1 - T_0)\frac{\partial \Theta}{\partial x}\Big|_{x=0} = -\frac{k(T_1 - T_0)}{\sqrt{4\alpha t}}\frac{d\Theta}{d\eta}\Big|_{\eta=0} = -\frac{k(T_1 - T_0)}{\sqrt{4\alpha t}}\left(-\frac{2}{\sqrt{\pi}}\right)$$

This leads us to the final result for the heat flux into the medium, which is

$$q_x|_{x=0} = \frac{k}{\sqrt{\pi\alpha t}}(T_1 - T_0)$$

We define the denominator as the thickness of the thermal boundary layer

$$\delta = \sqrt{\pi\alpha t}$$

Such that

$$q_x|_{x=0} = \frac{k}{\delta}(T_1 - T_0)$$

2.7.2 Estimating the Heat Transfer Coefficient

With the prior solution, we can approximate the heat transfer coefficient as

$$h = \frac{k}{\delta}$$

such that it scales with $k/\sqrt{\alpha t}$. Specifically, this applies when we are looking for the heat transfer coefficient on an object that can be treated using the fin approximation. As a result, it is important to know about values of k and α for common materials. For steel, $k = 60\,\frac{\mathrm{W}}{\mathrm{m\cdot K}}$ and $\alpha = 20\,\frac{\mathrm{mm^2}}{s}$. For water, $k = 0.6\,\frac{\mathrm{W}}{\mathrm{m\cdot K}}$ and $\alpha = 0.15\,\frac{\mathrm{mm^2}}{s}$. For air, $k = 0.02\,\frac{\mathrm{W}}{\mathrm{m\cdot K}}$ and $\alpha = 20\,\frac{\mathrm{mm^2}}{s}$. We can also generalize the results to say that metals have $k = O(10^2)$ and $\alpha = O(10^1) - O(10^{12})$, construction materials have $k = O(1)$ and $\alpha = O(1)$, water has $k = O(1)$ and $\alpha = O(10^{-1})$, and air has $k = O(10^{-2})$ and $\alpha = O(10^1)$ using the aforementioned units.

2.7.3 Ratios of Thermal Conductivity and Thermal Diffusivity

Now let us consider the scenario of touching two very different materials – steel and wood. We wish to explain why touching steel feels colder than wood even if both are at the same temperature. We will assume T_b is the temperature of the skin, T_o the temperature of the object, and T_i the temperature of the interface. We note that the heat flux must be equal on both sides of this interface. In addition, if we consider a sufficiently short initial, the solution can be modeled as diffusion from a flat boundary (we can ignore the curvature of either object). Then,

$$\frac{k_1}{\sqrt{\pi\alpha_1 t}}(T_b - T_i) = \frac{k_2}{\sqrt{\pi\alpha_2 t}}(T_i - T_o)$$

such that

$$\frac{k_1}{\sqrt{\alpha_1}}(T_b - T_i) = \frac{k_2}{\sqrt{\alpha_2}}(T_i - T_o)$$

If we define

$$\beta \equiv \frac{k_2/\sqrt{\alpha_2}}{k_1/\sqrt{\alpha_1}}$$

then

$$T_i = \frac{T_b + \beta T_o}{1+\beta}$$

As such, if $\beta \ll 1$ (e.g. for wood) then $T_i \approx T_b$. If $\beta \gg 1$ (e.g. for steel) then $T_i \approx T_o$.

2.8 Transient Heat Conduction

2.8.1 Negligible Internal Resistance

Recall that the transient heat conduction equation with a source term is given by

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T + \frac{\Phi}{\rho C_P}$$

We can break this down into two subcases.

The first case is when internal resistance is much smaller than the external resistance. For example, this may be for a highly conducting body immersed into a fluid. We first approximate the heat transfer coefficient as a constant, specifically

$$h = \frac{k}{L}$$

where L is some length characteristic of the body (usually the smallest length dimension). The fact that there is negligible internal resistance means that

$$h \ll \frac{k}{L}$$

which is the same as saying $\mathrm{Bi} \ll 1$. In this case, most of the temperature variation occurs in the fluid (in the boundary layer), while the temperature within the body is nearly spatially uniform, though changing with time. Then the surface temperature is nearly $\langle T \rangle$. This is the same type of problem we solved earlier where we had an object immersed in an infinite medium. The solution to that problem was simply

$$\Theta = e^{-\frac{t}{\tau}}$$

where

$$\tau \equiv \frac{\rho C_P V}{hA}$$

We see that the solution is not dependent on k. This is because it dropped out when we approximated $T \approx \langle T \rangle$ since that is equivalent to assuming $k \to \infty$ (no internal resistance).

2.8.2 Significant Internal Resistance

On the other hand, if there is significant internal resistance such that

$$h \gg \frac{k}{L}$$

then the system is well-described by the semi-infinite slab solution described before. This also means that the heat transfer coefficient can be approximated by

$$h = \frac{k}{\delta} = \frac{k}{\sqrt{\pi \alpha t}}$$

These two limiting cases are incredibly powerful in the study of heat transfer since they allow for the determination of heat transfer coefficients, which are not material properties.

2.9 Infinite Cylinder Paradox

Let us first consider a sphere of radius R with an internal heat source that is surrounded by fluid of infinite extent. The heat transfer equation we wish to use then is

$$\nabla^2 T = 0$$

if we are only considering the temperature of the fluid (not the temperature inside the sphere where the source term would matter). If we assume angular symmetry, then

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Theta}{\partial r}\right) = 0$$

if we define a dimensionless temperature difference of

$$\Theta = \frac{T - T_\infty}{T_s - T_\infty}$$

We specifically define the temperature difference in this way so that it has the proper limiting behavior of being zero at $r \to \infty$ and one at $r = R$. With this, we can integrate and find the solution to the temperature field in the fluid, which is

$$\Theta = \frac{R}{r}$$

The heat flux at the surface of the sphere can be found via Fourier's law to be

$$q_r|_{r=R} = -k \left.\frac{\partial T}{\partial r}\right|_{r=R} = -k(T_s - T_\infty) \left.\frac{\partial \Theta}{\partial r}\right|_{r=R}$$

We know from before that

$$\left.\frac{\partial \Theta}{\partial r}\right|_{r=R} = -\left.\frac{R}{r^2}\right|_{r=R} = -\frac{1}{R}$$

Therefore,

$$q_r|_{r=R} = \frac{k}{R}(T_s - T_\infty)$$

We then see that we can model the heat transfer coefficient in this problem via

$$h = \frac{k}{R}$$

The radius is then the appropriate unit of length to scale the thermal conductivity by to get the heat transfer coefficient. It should also be noted that k is out of the external medium (the fluid in this problem).

Now, we can try to repeat this problem for an infinite cylinder in a fluid. The problem is, it will be impossible to obtain an answer, just like how there is no solution for the analogous problem in fluid mechanics (called Stokes' paradox). The result of this paradox is that unlike most things, for an infinite cylinder, it is not the radial position away that influences how hot the fluid is but rather the distance away in *units of length* that make the difference. As an example, consider a needle. A needle can be modeled as an infinite cylinder since it has a length that is far larger than its thickness, so it appears infinite in the length dimension. If the needle is somehow heated up and put into a fluid, the fluid will be nearly as hot as the needle a few units of R away from the needle (this is not the case for the sphere problem, where we saw the heat flux decay with $1/R$). However, the fluid temperature will drop significantly just a few units of L away from the cylinder.

2.10 Order of Magnitude Analysis

We now wish to write an expression for δ using only an order of magnitude analysis and no other assumptions (so that we can make a statement beyond just the infinite slab problem). Ideally, we should get a similar solution. To do so, we start with the heat equation:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

We can model the first derivatives as the following over a sufficiently small distance $x = \delta$:

$$\frac{\partial T}{\partial t} \sim \frac{T_1 - T_0}{t}$$

$$\frac{\partial T}{\partial x} \sim \frac{T_1 - T_0}{\delta}$$

The second spatial derivative is easily obtainable from this information:

$$\frac{\partial^2 T}{\partial x^2} \sim \frac{1}{\delta}\left(\frac{\partial T}{\partial x}\right) \sim \frac{T_1 - T_0}{\delta^2}$$

Therefore, the heat equation can be approximated by

$$\frac{T_1 - T_0}{t} \approx \frac{\alpha(T_1 - T_0)}{\delta^2}$$

which simplifies to

$$\delta \sim \sqrt{\alpha t}$$

This is the same result as the infinite slab solution (except for the $\sqrt{\pi}$ term). This equation for the boundary layer is an incredibly important result of transport phenomena. In mass transfer, the equation is analogously

$$\delta \sim \sqrt{Dt}$$

2.11 Quasi Steady State Hypothesis

2.11.1 One-Dimensional Slab with Sinusoidal Boundary Temperature

Recall the problem of the one-dimensional slab with the solution

$$q_x = \frac{k}{L}(T_1 - T_2)$$

Now imagine that one of the boundary temperatures varies with time, such that

$$T_1 = T_2 + A\sin(\omega t)$$

Can we then plug in this expression into q_x and say that is the flux? The answer, generally speaking, is no. The solution is not at steady state and varies with time. However, there are limiting cases in which it is possible to use the linear profile as a reasonable approximation. The main criterion is that the time scale of change in external conditions is much greater than the internal relaxation time. In our problem, it is when

$$\frac{1}{\omega} \gg \frac{L^2}{\alpha}$$

where $1/\omega$ is the period of oscillation and $\tau \equiv L^2/\alpha$ is the internal relaxation time (obtained from the prior result of $\delta \sim \sqrt{\alpha t}$). If this condition applies, one can use the quasi-steady state assumption (QSSA) and use the linear profile.

2.11.2 Dissolution of Sphere into Fluid

Consider a solid sphere of initial radius a that dissolves in a large body of a fluid at rest. The mass concentration of the sphere material in the fluid in the state of equilibrium is ρ_{eq}. The value of ρ_{eq} is much less than ρ_s, where ρ_s is the mass density of the same material in the solid state. Find the time of complete dissolution of the sphere.

The characteristic time in the case of the sphere problem is $\tau = a^2/D$. This value will be much smaller than the time of dissolution under the QSSA. Let us define $b(t)$ as the radius of the sphere at a time t. Of course, $b \leq a$ for dissolution into the fluid. In the fluid (i.e. $r > a$ in spherical coordinates with the origin located at the center of the sphere), we have

$$\rho = \rho_{eq}\frac{b}{r}$$

This is analogous to the case of the heated sphere in the infinite medium, which had the solution $\Theta = R/r$. We now note that the flux is given by Fick's law as

$$\vec{j} = -D\,\text{grad}(\rho)$$

Therefore, at the interface ($r = b$),

$$\vec{j}|_{r=b} = -D\frac{\partial\rho}{\partial r}\bigg|_{r=b} = -D\rho_{eq}b\left(-\frac{1}{r^2}\right)\bigg|_{r=b} = \frac{D}{b}\rho_{eq}$$

We now must consider the shrinking of the solid sphere. We start with a mass balance:

$$\vec{j}\,dS\,dt = -\rho_s\,db\,dS$$

This then allows us to say that

$$\frac{db}{dt} = -\frac{\vec{j}}{\rho_s}$$

which, when plugging in $\vec{j}$, yields

$$\frac{db}{dt} = -\frac{D}{b}\frac{\rho_{eq}}{\rho_s}$$

Therefore, we can integrate from $t = 0$ to $t = t_{\text{diss}}$ and $b = a$ to $b = 0$ to arrive at

$$t_{\text{diss}} = \frac{a^2}{2D}\frac{\rho_s}{\rho_{eq}}$$

2.11.3 Diffusion with Irreversible Reaction

Assume that a species is consumed via fast, irreversible first-order reaction in a stagnant fluid. This means that our governing equation is

$$\frac{\partial c}{\partial t} = D\,\nabla^2(c) - k_1 c$$

We know that $c = c_b$ at the boundary (where c_b is the concentration in thermodynamic equilibrium with the concentration of the same species across the boundary), and $c = 0$ far from the boundary (at ∞). The solution is simple if the boundary is flat and extends to ∞ in y and z, or the flux in y and z is zero due to the presence of walls. The solution is also simple even if the boundary has an arbitrary shape, so long as the reaction is fast enough that the thickness of the boundary layer where the concentration is significant is much less than the radius of curvature of the boundary. Then,

$$0 = D\frac{\partial^2 c}{\partial x^2} - k_1 c$$

The boundary conditions are $c|_{x=0} = c_b$ and $c|_{x\to\infty} \to 0$. This then leads to

$$\frac{\partial^2 c}{\partial x^2} - \frac{c}{\lambda^2} = 0$$

where $\lambda \equiv \left(\frac{D}{k_1}\right)^{\frac{1}{2}}$, which can be integrated to yield

$$c = c_b e^{-\frac{x}{\lambda}}$$

with a flux at the surface of

$$j_x|_{x=0} = -D\frac{\partial c}{\partial x}\bigg|_{x=0} = \frac{D}{\lambda}c_b = \sqrt{Dk_1}c_b$$

2.11.4 Dissolution of Sphere with Irreversible Reaction

If we model the dissolution and reaction of a sphere in a fluid using the “flat Earth approximation”, we can use the same procedure as before. This means that the flux at the surface is

$$j|_{r=b} = \sqrt{Dk_1}\rho_{\text{eq}}$$

Then, to get the time of dissolution, we use the same mass balance as before of

$$\vec{j}\,dS\,dt = -\rho_s\,db\,dS$$

Plugging in the flux yields

$$\sqrt{Dk_1}\rho_{\text{eq}}dt = -\rho_s\,db$$

Therefore, we can integrate from $t = 0$ to $t = t_{\text{diss}}$ and $b = a$ to $b = 0$ to arrive at

$$t_{\text{diss}} = \frac{a}{\sqrt{Dk_1}} \frac{\rho_s}{\rho_{\text{eq}}}$$

The value of $\sqrt{D/k_1}$ is the thickness of the layer where diffusion and reaction occurs. If we wanted to solve the identical problem for a cylinder, we could in theory use the same procedure and answer if we once again use the "flat Earth approximation". Just diffusion (no reaction) from an infinite cylinder, however, has no solution just like in the heat conduction example.

2.12 Boundary Layer Theory

2.12.1 Overview

Consider flow over a flat plate. We will define

$$\Theta \equiv \frac{T - T_s}{T_a - T_s}$$

where $\Theta = 0$ at the plate surface and $\Theta = 1$ far from the surface. The temperature at the surface is T_a, and the temperature before the fluid comes into contact with the plate is T_a. Consider the fluid element crossing the plate at $x = 0$ and $t = 0$. The boundary layer thickness is simply the depth of penetration of heat into that fluid. Therefore,

$$\delta_T \sim \sqrt{\alpha t}$$

If we set $t = x/U$, then

$$\delta_T \sim \sqrt{\frac{\alpha x}{U}}$$

We then know that

$$h \sim \frac{k}{\delta_T} \sim k \left(\frac{U}{\alpha x}\right)^{\frac{1}{2}}$$

2.12.2 Heat Transfer to a Free-Surface in Flow

Instead of flow over a flat plate, we now consider a fluid in flow that then comes into contact with a well-mixed gas (instead of the plate). The interface has a temperature T_s, the fluid is moving at a constant velocity U, and the inlet temperature before meeting the gas is T_a. We begin with

$$\frac{DT}{Dt} = \alpha \nabla^2 T$$

which is

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \nabla T = \alpha \nabla^2 T$$

We note that $\vec{u} = U\hat{x}$ such that

$$\vec{u} \cdot \nabla T = U \frac{\partial T}{\partial x}$$

Therefore,

$$U\frac{\partial T}{\partial x} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

for the steady-state solution. In the limit of a very thin boundary layer (large U and small α),

$$\frac{\partial^2 T}{\partial x^2} \ll \frac{\partial^2 T}{\partial y^2}$$

such that

$$U\frac{\partial T}{\partial x} = \alpha\frac{\partial^2 T}{\partial y^2}$$

We now make a change of variables such that

$$\frac{\partial \Theta}{\partial x} = \frac{\alpha}{U}\frac{\partial^2 \Theta}{\partial y^2}$$

using the definition of Θ from before, where $\Theta = 0$ at $y = 0$, $\Theta \to 1$ at $y \to \infty$, and $\Theta = 1$ at $x = 0$. Mathematically, this is similar to the transient heat conduction in a semi-infinite slab but with t replaced with x/U. Using that solution (noting that I have the error function, not the complementary error function simply based on the slightly modified definition of Θ), we get

$$\Theta = \operatorname{erf}\left(\frac{y}{\sqrt{\frac{4\alpha x}{U}}}\right)$$

with

$$h = \frac{k}{\sqrt{\frac{\pi\alpha x}{U}}} = \frac{1}{\sqrt{\pi}}k\left(\frac{U}{\alpha x}\right)^{\frac{1}{2}}$$

This then of course implies that

$$\delta = \sqrt{\frac{\pi\alpha x}{U}}$$

We can calculate the average heat transfer coefficient by

$$\bar{h} = \frac{\int_0^L h\,dx}{L} = \frac{2}{\sqrt{\pi}}k\left(\frac{U}{\alpha L}\right)^{\frac{1}{2}}$$

We see that the numerical factor for $h \sim k\left(\frac{U}{\alpha x}\right)^{\frac{1}{2}}$ is $\frac{1}{\sqrt{\pi}} \approx 0.564$. This expression is true if we are considering the very simple case of uniform flow.

2.12.3 Flow over a Plate

If we apply the no-slip boundary condition, the exact solution has the following form

$$h = 0.33k\left(\frac{U}{\nu x}\right)^{\frac{1}{2}}\left(\frac{\nu}{\alpha}\right)^{\frac{1}{3}}$$

$$\bar{h} = 0.6k\left(\frac{U}{\nu L}\right)^{\frac{1}{2}}\left(\frac{\nu}{\alpha}\right)^{\frac{1}{3}}$$

applicable over the region $\nu/\alpha \geq 0.6$. It can be used for both $\mathrm{Pr} \approx 1$ and $\mathrm{Pr} \gg 1$ (as will be shown below), and provides a good interpolation for intermediate values.

Let us now consider how we can get the above functional form. Specifically, we will consider the case of $\mathrm{Pr} \gg 1$ but instead of a free-surface in constant flow, we have flow over a plate. The key difference here is that there is a no-slip boundary condition, and U is not constant everywhere. In this case, the hydraulic boundary layer will be significantly larger than the thermal boundary layer. The hydraulic boundary layer is given by

$$\delta_H \sim \sqrt{\frac{\nu x}{U}}$$

This U is just the upstream velocity, and is a reasonable approximation since far the velocity is approximately U a reasonable distance from the plate (where the velocity is zero due to no-slip). Since δ_T is smaller than δ_H, it does not extend as far out from the plate, and therefore the velocity in the thermal boundary layer is not the upstream velocity. Instead, we define a velocity in the thermal boundary layer as U_T, so

$$\delta_T \sim \sqrt{\frac{\alpha x}{U_T}}$$

We also note that the ratio of the boundary layers is approximately proportional to the ratio of the velocities within those boundary layers:

$$\frac{U_T}{U} \sim \frac{\delta_T}{\delta_H}$$

This is a key assumption of this mathematical development. It assumes that the velocity profile appears

Solving for U_T and plugging this into δ_T yields

$$\delta_T \sim \left(\frac{\alpha x}{U}\frac{\delta_H}{\delta_T}\right)^{\frac{1}{2}}$$

We now solve for δ_T to get

$$\delta_T^3 \sim \frac{\alpha x}{U}\ \delta_H$$

Plugging in for δ_H yields

$$\delta_T \sim \frac{1}{\mathrm{Pr}^{\frac{1}{3}}}\left(\frac{\nu x}{U}\right)^{\frac{1}{2}}$$

or

$$\delta_T \sim \frac{\delta_H}{\mathrm{Pr}^{\frac{1}{3}}}$$

With this, we can find our expression for the local-area heat transfer coefficient. We start at

$$h \sim \frac{k}{\delta_T}$$

to get

$$h \sim k\left(\frac{U}{\nu x}\right)^{\frac{1}{2}}\left(\frac{\nu}{\alpha}\right)^{\frac{1}{3}}$$

While we cannot get the numerical factory from this simplified approach, we do get the functional form that was shown earlier. Of course, as already stated, the numerical factor is 0.33 such that

$$h = 0.33k\left(\frac{U}{\nu x}\right)^{\frac{1}{2}}\left(\frac{\nu}{\alpha}\right)^{\frac{1}{3}}$$

We derived this expression for $\mathrm{Pr} \gg 1$, but when $\mathrm{Pr} \approx 1$, the equation can still be used since in that case $\delta_T \sim \delta_H$.

If we have $\mathrm{Pr} \ll 1$, the above expression cannot be used. However, the solution is still straight-forward because we can note that in the limit of low Prandtl numbers, the velocity in the thermal boundary layer looks like the upstream velocity. In that case, the solution is that of

$$h = \frac{1}{\sqrt{\pi}}k\left(\frac{U}{\alpha x}\right)^{\frac{1}{2}}$$

2.13 Green's Function Solution

Consider a thin film of hot water on the skin or holding a thin hot (or cold) metal sheet between your fingers. The excess heat in the film per unit area in the yz plane is

$$Q_A = \delta \rho_\delta C_{p_\delta}(T_\delta - T_0)$$

where Q is specifically the heat deposited per unit area. At any moment, except in the beginning, we must have

$$\int_{-\infty}^{\infty} \rho C_p (T - T_0)\, dx = Q_A = \text{constant}$$

We shall denote

$$\omega \equiv \frac{\rho C_p (T - T_0)}{Q_A}$$

which has units of inverse length. The initial condition is $\omega = 0$ at $t = 0$, $x \neq 0$. Idealizing $\delta \to 0$, we get $\omega \to \infty$ at $t = 0$, $x = 0$, making the above integral valid for all time. From this, ω satisfies the heat equation:

$$\frac{\partial \omega}{\partial t} = \alpha \frac{\partial^2 \omega}{\partial x^2}$$

The solution is

$$\omega(x,t) = \frac{1}{(4\pi\alpha t)^{\frac{1}{2}}} e^{-\frac{x^2}{4\alpha t}}$$

If we wish to scale this up to 2D or 3D, we can find the solution of ω everywhere by multiplying together each dimension's solution. Therefore, in 3D, the solution would have the following form:

$$\omega(x,y,z,t) = \frac{1}{(4\pi\alpha t)^{\frac{3}{2}}} e^{-\frac{(x^2+y^2+z^2)}{4\alpha t}}$$

but now ω would have units of inverse volume (i.e. it always has units of inverse the number of spatial dimensions it is a function of).

This can be used to find the temperature from

$$T = T_0 + \frac{Q_A}{\rho C_p} \omega$$

2.14 Heated Sphere in Flow

Consider a hot metal sphere, of radius a, with surface temperature T_s. It is suspended in a steam with a velocity U and upstream temperature T_0. Find the temperature distribution in the fluid, excluding the region in the vicinity of the sphere. We start by using the following relationship for the Nusselt number of a sphere:

$$\mathrm{Nu} = 2 + 0.6\mathrm{Re}^{\frac{1}{2}}\mathrm{Pr}^{\frac{1}{3}}$$

Recall that

$$\mathrm{Nu} \equiv \frac{hL}{k}$$

and so for a sphere, with $L = 2a$,

$$\mathrm{Nu} = \frac{2ha}{k}$$

This formula for the Nusselt number equals 2 for the situation of no flow, which comes about because $h = k/a$ for a sphere. We plug in the expression for the Nusselt number for a sphere into the above expression to get

$$h = \frac{k}{2a}\left(2 + 0.6\mathrm{Re}^{\frac{1}{2}}\mathrm{Pr}^{\frac{1}{3}}\right)$$

We want to find $\dot{Q}$, so we note

$$\dot{Q} = Aq = 4\pi a^2 h(T_s - T_0) = 2\pi a k\left(2 + 0.6\mathrm{Re}^{\frac{1}{2}}\mathrm{Pr}^{\frac{1}{3}}\right)(T_s - T_0)$$

We now write the heat equation

$$U\nabla T = \alpha_t \nabla^2 T$$

We can neglect the second-derivative in x (compared to the second-derivatives in y and z) if we assume we are dealing with large U (and the flow is in the x direction). In 2D, the solution is

$$\varphi = \frac{1}{4\pi\alpha t} e^{-\frac{(y^2+z^2)}{4\alpha_t t}}$$

We know that $t = x/U$ so

$$\varphi = \frac{U}{4\pi\alpha x} e^{-\frac{U(y+z^2)}{4\alpha_t x}}$$

We then recall that

$$\varphi = \frac{\rho C_P(T - T_0)}{\text{heat deposited per unit length}} = \frac{\rho C_p(T - T_0)}{Q_L}$$

We can relate Q_L to our $\dot{Q}$ by

$$Q_L = \frac{\dot{Q}}{U}$$

such that

$$\varphi = \frac{U\rho C_p(T - T_0)}{\dot{Q}}$$

Rearranging this in terms of T yields the solution for the temperature field, with φ defined as above:

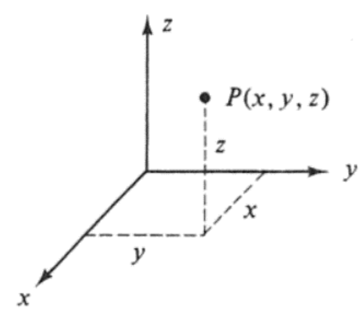
$$T = T_0 + \frac{\dot{Q}}{U\rho C_p}\varphi$$

3 APPENDIX: VECTOR CALCULUS

3.1 COORDINATE SYSTEMS

3.1.1 CARTESIAN COORDINATE SYSTEM

The following diagram is a schematic of the Cartesian coordinate system.

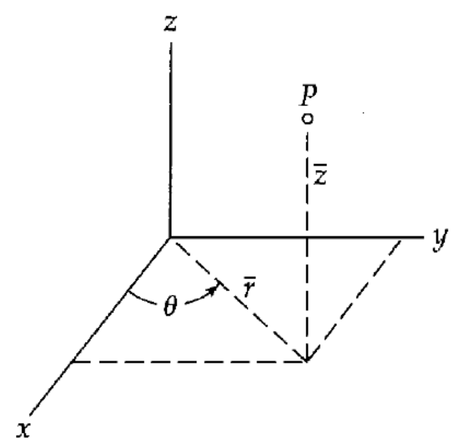


With this definition, the position vector in Cartesian coordinates is

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

3.1.2 CYLINDRICAL COORDINATE SYSTEM

The following diagram is a schematic of the cylindrical coordinate system. Take note that the standard definition is that the sign of the azimuth is considered positive in the counter clockwise direction.



With this definition, the position vector in cylindrical coordinates is

$$\vec{r} = r\hat{r} + z\hat{z}$$

To convert from cylindrical coordinates to Cartesian coordinates,

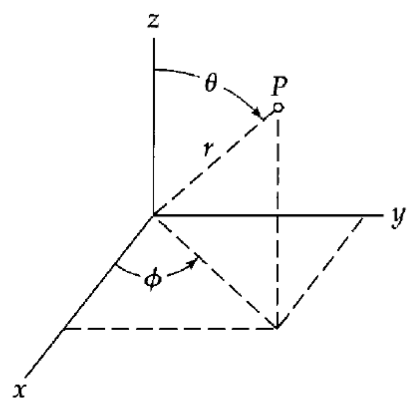
$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

3.1.3 SPHERICAL COORDINATE SYSTEM

The following diagram is a schematic of the spherical coordinate system. Note that many mathematics textbooks use a slightly different convention by swapping the definitions of θ and ϕ. Take note that the standard definition is that the sign of the azimuth is considered positive in the counter clockwise direction and that the inclination angle is the angle between the zenith direction and a given point.



With this definition, the position vector in spherical coordinates is

$$\vec{r} = r\hat{r}$$

To convert from spherical coordinates to Cartesian coordinates,

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

3.1.4 SURFACE DIFFERENTIALS

The surface differentials, dS, in each of the three major coordinate systems are as follows.

Coordinate system	Surface differential, dS
Cartesian (top, $\hat{n} = \hat{z}$)	$dx\ dy$
Cartesian (side, $\hat{n} = \hat{y}$)	$dx\ dz$
Cartesian (side, $\hat{n} = \hat{x}$)	$dy\ dz$
Cylindrical (top, $\hat{n} = \hat{z}$)	$r\ dr\ d\theta$
Cylindrical (side, $\hat{n} = \hat{r}$)	$r\ d\theta\ dz$
Spherical ($\hat{n} = \hat{r}$)	$r^2\sin\theta\ \ d\theta\ d\phi$

3.1.5 VOLUME DIFFERENTIALS

The volume differentials, dV, in each of the three major coordinate systems are as follows.

Coordinate system	Volume differential, dV
Cartesian	$dx\ dy\ dz$
Cylindrical	$r\ dr\ d\theta\ dz$
Spherical	$r^2\sin\theta\ \ dr\ d\theta\ d\phi$

3.2 OPERATORS

3.2.1 GRADIENT

The gradient is a mathematical operator that acts on a scalar function and is written as $\mathrm{grad}(f)$ or ∇f. The result is always a vector. It is essentially the derivative applied to functions of several variables.

In Cartesian coordinates, the gradient is

$$\mathrm{grad}(f) = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

In cylindrical coordinates, the gradient is

$$\mathrm{grad}(f) = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{\partial f}{\partial z}\hat{z}$$

In spherical coordinates, the gradient is

$$\mathrm{grad}(f) = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}$$

3.2.2 DIVERGENCE

The divergence is a mathematical operator that acts on a vector function and is written as $\mathrm{div}(\vec{v})$ or $\nabla \cdot \vec{v}$. The result is always a scalar. The divergence represents the flux emanating from any point of the given vector function (essentially, a rate of loss of a specific quantity).

In Cartesian coordinates, the divergence is

$$\mathrm{div}(\vec{v}) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

In cylindrical coordinates, the divergence is

$$\mathrm{div}(\vec{v}) = \frac{1}{r}\frac{\partial}{\partial r}\left(r v_r\right) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

In spherical coordinates, the divergence is

$$\mathrm{div}(\vec{v}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(v_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

3.2.3 CURL

The curl is a mathematical operator that acts on a vector function and is written as $\mathrm{curl}(\vec{v})$ or $\nabla \times \vec{v}$. The result is always a vector. The curl represents the infinitesimal rotation of a vector function.

In Cartesian coordinates, the curl is

$$\mathrm{curl}(\vec{v}) = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{y} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{z}$$

In cylindrical coordinates, the curl is

$$\operatorname{curl}(\vec{v}) = \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)\hat{r} + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial (r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta}\right)$$

In spherical coordinates, the curl is

$$\operatorname{curl}(\vec{v}) = \frac{1}{r\sin\theta}\left(\frac{\partial (v_\phi \sin\theta)}{\partial \theta} - \frac{\partial v_\theta}{\partial \phi}\right)\hat{r} + \left(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{1}{r}\frac{\partial (r v_\phi)}{\partial r}\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial (r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta}\right)\hat{\phi}$$

3.2.4 LAPLACIAN

The Laplacian is a mathematical operator that acts on a scalar function and is written as $\nabla^2 f$. The result is always a scalar. It represents the divergence of the gradient of a scalar function.

In Cartesian coordinates, the Laplacian is

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

In cylindrical coordinates, the Laplacian is

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

In spherical coordinates, the Laplacian is

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

3.2.5 MEMORIZATION SHORTCUTS

It can be a bit of a challenge to memorize the above equations for non-Cartesian coordinate systems. A shortcut can be used to memorize them in the special case where dependence is only on the r coordinate. To do so, we first define n as the number of angular coordinates (i.e. $n = 1$ for cylindrical and $n = 2$ for spherical). Then,

$$\operatorname{grad}(f) = \frac{\partial f}{\partial r}\,\hat{r}$$

$$\operatorname{div}(\vec{\varphi}) = \frac{1}{r^n}\frac{\partial}{\partial r}(r^n \varphi_r)$$

$$\nabla^2 f = \frac{1}{r^n}\frac{\partial}{\partial r}\left(r^n\frac{\partial f}{\partial r}\right)$$

3.3 COMMON VECTOR CALCULUS IDENTITIES

The following are useful identities in vector calculus for a scalar field f and a vector field $\vec{\varphi}$.

The product of a scalar and vector is as follows:

$$\operatorname{div}(f\vec{\varphi}) = f\operatorname{div}(\vec{\varphi}) + \vec{\varphi}\cdot\operatorname{grad}(f)$$

Useful second derivative identities are shown below:

$$\operatorname{div}(\operatorname{grad}(f)) = \nabla^2 f$$

$$\operatorname{curl}(\operatorname{grad}(f)) = 0$$

$$\operatorname{div}(\operatorname{curl}(\vec{\varphi})) = 0$$

$$\nabla^2 \vec{\varphi} = \operatorname{grad}(\operatorname{div}(\vec{\varphi})) - \operatorname{curl}(\operatorname{curl}(\vec{\varphi}))$$

3.4 Surface Integration

3.4.1 The Surface Integral

The surface integral is a generalization of multiple integrals to integration over surfaces. It is the two-dimensional extension of the one-dimensional line integral. The notation of the surface integral is not agreed upon. Some texts using a double integral with an S beneath to indicate a surface integral, whereas other texts use the symbol for a line integral – an integral with a circle around the center – to represent surface integrals as well. Some other texts using a double integral with a circle around it. They all mean the same thing.

The surface integral of a scalar field is written and computed as

$$F = \oiint f \, dS$$

The surface integral of a vector field cannot be as easily computed. If one wants to compute the surface integral of, say, the force (which is a vector), one needs to convert it first to a scalar and then apply the direction at the end of the computation. As such, the general method of doing the surface integral of a vector is to say

$$F = \oiint \vec{f} \cdot \hat{k} \, dS$$

where $\hat{k}$ is in the same direction as F is anticipated to be in. In the special case of $\hat{k} = \hat{n}$, this surface integral is called the flux

$$\text{Flux} = \oiint \vec{f} \cdot \hat{n} \, dS$$

To make the computation of surface integrals easier, common systems and their corresponding dS equivalents are included in section 1.1.4. You can then simply substitute in for the surface element dS in the integral to convert it to a standard double integral and then apply the appropriate bounds.

3.4.2 Divergence Theorem

The divergence theorem can convert a surface integral into a volume integral when applied to a vector field via

$$\oiint \vec{f} \cdot \hat{n} \, dS = \iiint \operatorname{div}(\vec{f}) \, dV$$

The volume integral can be computed by substituting in the appropriate volume element dV and including the appropriate bounds.