FLUID MECHANICS

Based on CHEM_ENG 421 at Northwestern University

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1 NEWTON'S LAWS

1.1 INERTIAL FRAMES

Newton's first law states that velocity, $\vec{v}$, is a constant if the force, $\vec{F}$, is zero. Newton's second law is the very famous $\vec{F} = m\vec{a}$. At first glance, it would seem that Newton's first law is simply a recapitulation of the second law. After all, since acceleration, $\vec{a}$, is simply the first time derivative of velocity, then if velocity is constant, acceleration is zero and thereby force is zero. However, there is indeed a reason to explicitly state Newton's first law. Newton's first law sets the frame of reference as the inertial frame. Examples of nearly inertial frames are the Earth, an Earth-bound lab, and a train moving with constant speed with respect to the Earth.

Any rapidly rotating frame is a non-inertial reference frame. Non-inertial but accelerating frames rely on Einstein's equivalence principle, which states that

> "All the phenomena in a frame that are accelerating with respect to an inertial frame with acceleration $\vec{a}_f$ happens as if in an inertial frame with apparent gravity, where the acceleration of gravity is given by $-\vec{a}_f$."

1.2 NEWTON'S LAWS

In more exact terms, the first law can be said to mean:

> "There exists a frame of reference such that in this frame a body not acted upon by any force continues to be either in the state of rest or a uniform motion (i.e. with constant velocity). Such a frame is called inertial. Any frame moving with constant velocity with respect to an inertial frame is also inertial."

This should make sense and establishes a context for Newton's second law. Newton's second law can be effectively worded as "In an inertial frame, a body of mass m acted upon by a force $\vec{F}$, acquires an acceleration $\vec{a} = \vec{F}/m$."

1.3 GALILEAN INVARIANCE

The last sentence of Newton's first law in the prior subsection is a little more nuanced than it appears. Let's prove that for any frame moving with uniform motion with respect to an inertial frame is also inertial. This is frequently called the Galilean invariance.

Consider two inertial frames given by S and S' that both share a universal time. Suppose S' is in relative uniform motion to S with speed v. Therefore, for a position $r'(t)$ in S' frame and position $r(t)$ in S frame then

$$r'(t) = r(t) + vt$$

The velocity of the object in the S' frame is given by

$$u'(t) = \frac{dr'(t)}{dt} = \frac{d(r(t) + vt)}{dt} = \frac{dr(t)}{dt} + v = u(t) + v$$

Differentiating once more to yield acceleration gets

$$a'(t) = \frac{du'(t)}{dt} = \frac{d(u(t) + v)}{dt} = \frac{du(t)}{dt} = a(t)$$

Therefore, assuming the mass is identical in both inertial frames, Newton's laws in frame S should also be true in frame S' (and all other frames move with uniform relative motion to S).

2 MATHEMATICAL DESCRIPTION OF FLUID FLOW

2.1 FIELDS AND FORCES

Fluids can be described based on velocity vector fields, $\vec{u}(\vec{r},t)$, and pressure scalar fields $P(\vec{r},t)$. These variables satisfy differential equations, which express the two basic laws of nature: the conservation of mass and Newton's second law. The conservation of mass states that fluid can move from point to point, but it cannot be created or destroyed. Newton's second law implies that we use an inertial frame of reference; otherwise, fictitious forces such as centrifugal and Coriolos forces must be included.

Two kinds of forces are typically considered in the study of fluid mechanics. The first is long-range "body forces" such as gravity, usually known per unit mass ($\vec{g}$) or per unit volume (e.g. $\rho\vec{g}$). The second is surface "contact forces" due to the short-range action of fluids on fluids (or solids) across an imaginary (or real) interface, such as pressure or viscous friction.

2.2 CONTINUITY EQUATION

The conservation of mass for a fluid, and by extension the continuity equation, will be derived below.

1. Let's assume an arbitrary control volume in space, given by V. It has a surface S and a normal direction given by $\vec{n}$.
2. Since mass is conserved, we can say that
$$(\text{total rate of increase of mass in } V) = (\text{total net flow of mass into } V)$$
3. In integral form, this can be written as
$$\iiint (\text{rate of increase of mass in unit volume})\, dV$$
$$= \oiint (\text{amount of mass crossing, per unit time, a unit area of } S \text{ in the inward direction})\, dS$$
This can be mathematically expressed as
$$\iiint \frac{\partial\rho}{\partial t} dV = -\oiint \rho\vec{v}\cdot\hat{n}\, dS$$
4. Applying the divergence theorem (see Appendix) yields
$$\iiint \frac{\partial\rho}{\partial t} dV = -\iiint \nabla\cdot\rho\vec{v}\, dV$$
5. Therefore, we can say
$$\iiint \left(\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\vec{v}\right) dV = 0$$
6. The integrand must in and of itself be equal to zero because the above expression must be true for an arbitrary volume V and for any arbitrary integral bounds (i.e. for all volumes). As such, we arrive at the continuity equation
$$\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\vec{v} = 0$$

2.3 THE STRESS TENSOR

Consider $\vec{f}$ as the surface force per unit area of an imaginary (or real) interface dividing a fluid, exerted by the fluid "outside" (where $\vec{n}$ points) on the fluid (or solid) "inside". For a given point in space $\vec{r}$, movement of time t, and orientation $\vec{n}$ this force is a vector. However, $\vec{f}$ is not a vector field because it depends on $\vec{n}$ and t. Consider the simplest case: a fluid at rest with $\vec{u}(\vec{r},t) = 0$. Then, $\vec{f}$ is created by pressure only and is directed opposite to $\vec{n}$, and its magnitude is simply ρ.

Therefore, for this case

$$\vec{f} = -P\vec{n}$$

Since $\vec{f}(\vec{r},t,\vec{n}) = -P(\vec{r},t)\vec{n}$, this means that $\vec{f} = (\text{scalar field})\vec{n}$. But in the general case, when $\vec{u} \neq 0$ and $\vec{f}$ can be directed in all possible ways relative to $\vec{n}$, we need a new type of field: a tensor field, where the tensor is a second-rank invariant object (scalars being of zero rank, and vectors being of first rank) which has nine components. Then, in general

$$\vec{f} \equiv \bar{\bar{\sigma}} \cdot \vec{n}$$

where $\bar{\bar{\sigma}} = \bar{\bar{\sigma}}(\vec{r},t)$ is the stress tensor and depends only on the point in space $\vec{r}$ and time t, as a field should. It completely specifies the force distribution in a moving fluid due to contact forces.

In any given coordinate system, with unit vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ the components of a tensor make up a matrix given by

$$\bar{\bar{\sigma}} = \begin{pmatrix} \sigma_{ii} & \sigma_{ij} & \sigma_{ik} \\ \sigma_{ji} & \sigma_{jj} & \sigma_{jk} \\ \sigma_{ki} & \sigma_{kj} & \sigma_{kk} \end{pmatrix}; \quad \sigma_{ij} = \hat{\imath} \cdot \bar{\bar{\sigma}} \cdot \hat{\jmath}$$

In the particular case of a system at rest, $\vec{u} = 0$, the stress tensor $\bar{\bar{\sigma}}$ must be such that

$$\bar{\bar{\sigma}} \cdot \vec{n} = -P\vec{n}$$

for any $\vec{n}$. That is, $\bar{\bar{\sigma}} \cdot \vec{n} \propto \vec{n}$. This can be true for an arbitrary $\vec{n}$ only if $\bar{\bar{\sigma}} \propto \bar{\bar{\mathrm{I}}}$ where $\bar{\bar{\mathrm{I}}}$ is the identity tensor (also called unit tensor), which in all coordinate systems has the components

$$\bar{\bar{\mathrm{I}}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

As such,

$$\mathrm{I}_{ij} = \delta_{ij}$$

where δ_{ij} is the Kronecker delta. So, in a fluid at rest,

$$\bar{\bar{\sigma}} = -P\bar{\bar{\mathrm{I}}}$$

If $\vec{u} \neq 0$ (and $\vec{u} \neq \text{constant}$) then

$$\bar{\bar{\sigma}} = -P\bar{\bar{\mathrm{I}}} + \bar{\bar{\tau}}$$

where $\bar{\bar{\tau}}$ is sometimes called “extra stress” or more commonly “viscous stress”. It is nonzero if the fluid is sheared or strained. An easier way to use this tensor is to note that

$$\sigma_{ij} = -P\delta_{ij} + \tau_{ij}$$

where

$$\tau_{ij} = \mu\left(\nabla_i u_j + \nabla_j u_i\right)$$

2.4 NEWTON'S LAW OF VISCOSITY

When a simple fluid is sheared, it resists with the force (per unit area of the plane) which is proportional to the gradient (i.e. derivative) of velocity. For general motion, this becomes

$$\bar{\bar{\tau}} = 2\mu\bar{\bar{E}}$$

where μ is the viscosity and $\bar{\bar{E}}$ is the rate of strain tensor defined by

$$\bar{\bar{E}} \equiv \frac{1}{2}\left(\nabla\vec{u} + \nabla\vec{u}^{\mathrm{T}}\right)$$

where the superscript T refers to the transpose. Therefore,

$$\bar{\bar{\sigma}} = -P\bar{\bar{\mathrm{I}}} + 2\mu\bar{\bar{E}}$$

which is an approximate empirical relationship of Newtonian fluids, not valid for complicated fluids like polymer solutions. Note that $\bar{\bar{E}}$ is symmetric such that $E_{ij} = E_{ji}$. The term $\nabla\vec{u}$ is *not* the divergence of $\vec{u}$ since that would require a dot product. Rather, it represents the velocity gradient tensor, which is the gradient operator applied to $\vec{u}$. In Cartesian coordinates this is

$$\nabla\vec{u} = \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{pmatrix}$$

There is a point of notation that should be discussed. We should note that multiplying two tensors can be done via $\left(\vec{a}\vec{b}\right)_{ij} = a_i b_j$. This is also written as $\left(\vec{a}\otimes\vec{b}\right)_{ij} = a_i b_j$. The $\otimes$ operator is often referred to as the tensor product.

2.5 GENERALIZED GAUSS' THEOREM

Recall that the standard divergence theorem states that

$$\oiint \vec{f}\cdot\hat{n}\, dS = \iiint \mathrm{div}(\vec{f})\, dV$$

We can write a generalized Gauss' theorem that is as follows

$$\iiint \nabla(\text{operation})(\text{tensor})\, dV = \oiint \hat{n}(\text{operation})(\text{tensor})\, dS$$

Recall that a scalar is a tensor of rank 0, a vector is a tensor of rank 1, and then tensors we have described here are rank 2. The (operation) term can be a variety of things, such as (dot product), (ordinary product), (tensor product), or (cross product). In the case of (operation) = (dot product) and (tensor) = (vector) then we arrive at the divergence theorem.

Let us consider an example of the generalized Gauss' theorem. In this case, we will have (operation) = (ordinary product) and (tensor) = (scalar). Therefore,

$$\int \nabla f\, dV = \oint \hat{n} f\, dS$$

In one-dimensional space between the points a and b, the gradient operator is just $\nabla = \hat{x}\frac{d}{dx}$ for a line in the x dimension. We also note that $dV = dx$ for this system. The lefthand term then becomes

$$\int \nabla f \, dV = \int_a^b \hat{x}\frac{df}{dx}dx = \int_a^b \hat{x}df$$

Since there is no surface to integrate over, the righthand term becomes

$$\oint \hat{n}f \, dS = (f|_{x=b} - f|_{x=a})\hat{x}$$

Therefore, equation the two expressions (and dropping the $\hat{x}$ since we only have one dimension anyway)

$$\int_a^b df = f|_{x=b} - f|_{x=a}$$

which is the fundamental theorem of calculus!

2.6 NEWTON'S EQUATION OF MOTION FOR A FLUID

Recall that in an inertial frame, $\vec{F} = m\vec{a}$. This will apply to any fluid element so small that it can be considered to have a single value of $\vec{a}$. Let it be infinitesimally small. Then, $m = \rho \, dV$ and $\vec{F} = \vec{\phi} \, dV$, where $\vec{\phi}$ is the total (body and contact) force per unit volume. Then $\rho\vec{a} = \vec{\phi}$.

If only gravity were present, then $\vec{\phi} = \rho\vec{g}$, but we need to find contact force per unit volume given the stress tensor field $\bar{\bar{\sigma}}$. Consider a continuum (e.g. fluid) at rest ($\vec{u} = 0, \vec{a} = 0$) under the combined action of some arbitrary body force field (not only gravity) and $\bar{\bar{\sigma}}$. Then, $0 = \vec{\psi} = \vec{\psi}_b + \vec{\psi}_c$ where the b subscript is for body force and c subscript is for contact force. We also have that $\vec{\psi}_b(\vec{r})$ is known (imposed from things such gravity, electromagnetic field, etc.) while $\vec{\psi}_c(\vec{r})$ depends only on $\bar{\bar{\sigma}}(\vec{r})$.

Any part of this continuum, enclosed in a volume V, is at rest, and so the total force acting on it must be zero. Thus,

$$(\text{total body force in } V) + (\text{sum of contact forces acting on matter in } V \text{ across surface } S) = 0$$

This becomes

$$\iiint \vec{\psi}_b \, dV + \oint\!\!\oint \vec{f} \, dS = 0$$

Substituting in for the force

$$\iiint \vec{\psi}_b \, dV + \oint\!\!\oint \bar{\bar{\sigma}} \cdot \hat{n} \, dS = 0$$

Applying the divergence theorem

$$\iiint \vec{\psi}_b \, dV + \iiint \nabla \cdot \bar{\bar{\sigma}} \, dV = 0$$

Combing the integrals

$$\iiint [\vec{\psi}_b + \nabla \cdot \bar{\bar{\sigma}}]\, dV = 0$$

Like with the continuity equation, this expression must hold for an arbitrary volume, and therefore the integrand itself must be zero, such that

$$\vec{\psi}_b + \nabla \cdot \bar{\bar{\sigma}} = 0$$

We can now introduce motion by making the expression no longer equal zero. Since it is a force balance, we can say that

$$\rho\vec{a} = \vec{\psi}_b + \nabla \cdot \bar{\bar{\sigma}}$$

In the absence of electromagnetic effects, $\vec{\psi}_b = \rho\vec{g}$, so

$$\rho\vec{a} = \rho\vec{g} + \nabla \cdot \bar{\bar{\sigma}}$$

This is known as Newton's Second Law for fluids. We can write $\vec{a} = d\vec{u}/dt$ of the fluid element, so

$$\rho\frac{D\vec{u}}{Dt} = \rho\vec{g} + \nabla \cdot \bar{\bar{\sigma}}$$

This equation is exact with no assumptions other than Newton's Second Law.

2.7 NAVIER-STOKES EQUATION

We can now derive the Navier-Stokes equation from Newton's Second Law for fluids. We make two key simplifications. For any arbitrary scalar field $f(\vec{r})$,

$$\nabla \cdot f\bar{\bar{I}} = \nabla f$$

For an incompressible fluid

$$\nabla \cdot \left(\nabla\vec{u} + \nabla\vec{u}^{\mathrm{T}}\right) = \nabla^2\vec{u}$$

This then means that

$$\nabla \cdot \bar{\bar{\sigma}} = -\nabla P + \mu\nabla^2\vec{u}$$

Therefore, we arrive at

$$\rho\frac{D\vec{u}}{Dt} = -\nabla P + \mu\nabla^2\vec{u} + \rho\vec{g}$$

where the capital D indicates a material derivative. This is defined as

$$\frac{D\vec{u}}{Dt} \equiv \frac{\partial\vec{u}}{\partial t} + \frac{\partial\vec{r}}{\partial t}\frac{\partial\vec{u}}{\partial\vec{r}} = \frac{\partial\vec{u}}{\partial t} + \vec{u} \cdot \nabla\vec{u}$$

As such, the Navier-Stokes equation (for an incompressible fluid) simplifies to

$$\rho\left(\frac{\partial\vec{u}}{\partial t} + \vec{u} \cdot \nabla\vec{u}\right) = -\nabla P + \mu\nabla^2\vec{u} + \rho\vec{g}$$

2.8 SIMPLIFYING THE NAVIER-STOKES EQUATION

2.8.1 REDUCED PRESSURE

The Navier-Stokes equation can often be written without explicit use of gravity terms through the use of a modified pressure, P_{mod}. Modified pressure can be used when the problem does not involve free surfaces (e.g. water/air interface). The modified pressure is defined as

$$P_{\text{mod}} \equiv P - P_{\text{hydrostatic}}$$

where

$$P_{\text{hydrostatic}} = \rho\vec{g} \cdot \vec{r} + \text{constant}$$

Then,

$$P = P_{\text{mod}} + \rho\vec{g} \cdot \vec{r} + \text{constant}$$

With this, the $\rho\vec{g} - \nabla P$ term in the N-S equation becomes

$$\rho\vec{r} - \nabla P = \rho\vec{g} - \nabla P_{\text{mod}} - \rho\nabla(\vec{g} \cdot \vec{r} + \text{constant}) = -\nabla P_{\text{mod}}$$

From this, we can see that we can neglect the gravity term in the N-S equation and replaced ∇P with ∇P_{mod}. This means that

$$\rho\left(\frac{\partial\vec{u}}{\partial t} + \vec{u} \cdot \nabla\vec{u}\right) = -\nabla P_{\text{mod}} + \mu\nabla^2\vec{u}$$

Oftentimes, the “mod” subscript is omitted for brevity’s sake.

2.8.2 REYNOLDS NUMBER

The Navier-Stokes equation can be written using only dimensionless quantities. A dimensionless variable is defined as the original value divided by a given scale. For distances, you should use an appropriate length given the boundary conditions and is denoted L. The dimensionless velocity is denoted u. The dimensionless pressure is simply ρu^2, and the dimensionless time is then L/u. With these definitions, the Navier-Stokes equation becomes the following

$$\frac{\partial\vec{u}}{\partial t} + \vec{u} \cdot \nabla\vec{u} = -\nabla P + \frac{1}{\text{Re}}\nabla^2\vec{u}$$

where Re is the Reynolds number is

$$\text{Re} \equiv \frac{\rho u L}{\eta} = \frac{uL}{\nu}$$

If the geometry of two problems is similar except for scale, and the Reynolds number is identical in both cases, then the mathematical solutions, in scaled dimensionless variables, are identical. This is the basis behind most “scale-up” studies.

2.8.3 APPROXIMATE SOLUTIONS OF THE NAVIER-STOKES EQUATION

Generally, the forces on a given fluid are

$$-(\text{inertia force}) = (\text{pressure force}) + (\text{viscous force})$$

where inertia force is simply $-m\vec{a}$. In many flows, the pressure force is mainly balanced by either the inertial force or viscous force. The key question is when either of these terms can be neglected in the Navier-Stokes equation. We will do this by estimating the ratio of

$$\frac{\left|\rho \frac{D\vec{u}}{Dt}\right|}{|\eta \nabla^2 \vec{u}|}$$

After all of the relevant quantities are made dimensionless, this ratio becomes the following (assuming steady flow)

$$\mathrm{Re}\frac{\left|\frac{D\vec{u}}{Dt}\right|}{|\nabla^2 \vec{u}|} = \mathrm{Re}\frac{|\vec{u} \cdot \nabla v|}{|\nabla^2 \vec{u}|}$$

If $\mathrm{Re} \ll 1$, viscous forces are dominant, which is called Stokes flow, creeping flow, or low Reynolds flow. This then means that the Navier-Stokes equation can be written as

$$0 = -\nabla P + \eta \nabla^2 \vec{u}$$

where the left-hand side of the Navier-Stokes equation is approximately zero under this assumption. Note that this drops out the density term.

3 FORCES AND TORQUES: SPHERE IN STOKES FLOW

3.1.1 PROBLEM SETUP

Consider Stokes flow past a sphere of radius a. Far upstream, the flow is uniform with velocity U, and the pressure there is P_0. In the spherical coordinate system centered at the center of the sphere, the axis $\theta = 0$ is along the direction of the incoming flow (which will be said to flow in the $\hat{k}$ direction). It can be derived (although it will simply be stated here) that the velocity profiles and pressure profile are

$$u_r = U\left(1 - \frac{3}{2}\frac{a}{r} + \frac{1}{2}\left(\frac{a}{r}\right)^3\right)\cos\theta$$

$$u_\theta = -U\left(1 - \frac{3}{4}\frac{a}{r} - \frac{1}{4}\left(\frac{a}{r}\right)^3\right)\sin\theta$$

$$u_\phi = 0$$

and

$$P = P_0 - \frac{3}{2}\frac{\eta U}{a}\left(\frac{a}{r}\right)^2\cos\theta$$

The goal is to calculate the total force on the sphere from the fluid, which can be calculated as the sum of the force due to the tangential stress on the surface of the sphere, the force from the viscous normal stress, and the force on the sphere due to pressure. Why is this so? First, let's write out the expression for force:

$$\vec{f} = \bar{\bar{\sigma}} \cdot \hat{n}$$

We know that the normal direction on the surface of the sphere is $\hat{r}$, so $\hat{n} = \hat{r}$. We can rewrite the force as

$$\vec{f} = \bar{\bar{\sigma}} \cdot \hat{r}$$

From symmetry, we can make the argument that the force has no ϕ component. Therefore,

$$\vec{f} = f_r\hat{r} + f_\theta\hat{\theta}$$

where $f_r \equiv \hat{r} \cdot \vec{f}$ and is the normal stress and $f_\theta \equiv \hat{\theta} \cdot \vec{f}$ and is the tangential stress. We can now state that

$$f_r = \hat{r} \cdot \vec{f} = \hat{r} \cdot \bar{\bar{\sigma}} \cdot \hat{r} = \sigma_{rr} = -P\delta_{rr} + \tau_{rr} = -P + 2\eta\frac{\partial u_r}{\partial r}$$

We can also state that

$$f_\theta = \hat{\theta} \cdot \vec{f} = \hat{\theta} \cdot \bar{\bar{\sigma}} \cdot \hat{r} = \sigma_{\theta r} = -P\delta_{\theta r} + \tau_{\theta r} = \eta\left(r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)$$

From this, we see that there are three components to the total force: the force due to tangential stress $\tau_{\theta r}$, the force due to normal stress τ_{rr}, and the force due to pressure.

3.1.2 FORCE DUE TO TANGENTIAL STRESS

We now look at calculating the force due to the tangential stress, $\tau_{\theta r}$. Therefore, we must first calculate the tangential stress, which turns out to be

$$\tau_{\theta r} = \eta\left(r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right) = -\frac{3}{2}\frac{\eta U \sin\theta}{a}$$

The force due to tangential stress is then

$$\vec{F} = \oiint f_\theta \hat{\theta}\, dS = \oiint \tau_{\theta r}\, \hat{\theta} dS$$

It is difficult to do the surface integral of a vector, so some simplification must be made. By symmetry, we can note that the force on the sphere will only act in the direction of flow (which we have defined as the $\hat{k}$ direction). Therefore, we can say multiply both sides by $\hat{k}$

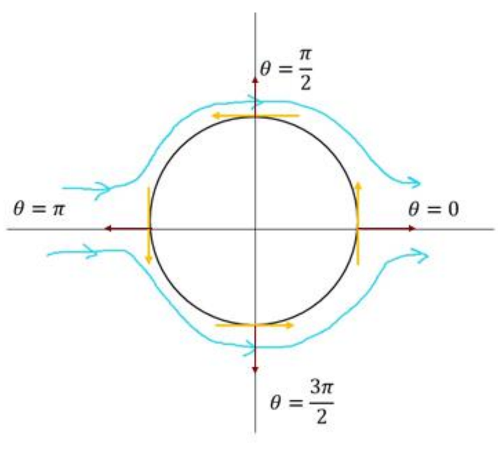
$$\vec{F} \cdot \hat{k} = \left(\oiint \tau_{\theta r}\, \hat{\theta} dS\right) \cdot \hat{k}$$

Since $\hat{k}$ is a constant, it can be pulled into the integrand (and we can note that $F = \vec{F} \cdot \hat{k}$)

$$F = \oiint \tau_{\theta r}\, \hat{\theta} \cdot \hat{k} dS$$

We then note that $\hat{\theta} \cdot \hat{k} = -\sin\theta$. This is not a trivial statement, so let's understand this a bit more.

The figure on the next page is a 2D projection of the 3D sphere, where the cyan arrows represent the fluid, and the yellow arrows represent the tangential force (the maroon arrows represent the normal force, which we will return to later). Due to mathematical convention (and the right-hand rule), the quadrants are numbered counter clockwise such that the angles are as shown below (note that we defined in the problem statement that $\theta = 0$ aligns with the direction of the fluid flow). We see that at $\theta = 0$ and $\theta = \pi$, the force in the $\hat{k}$ direction should be zero. We also see that at $\theta = \pi/2$, the force should be in the $-\hat{k}$ direction whereas at $\theta = 3\pi/2$, the force should be in the $+\hat{k}$ direction. The trigonometric function that satisfies these conditions is $-\sin\theta$ and therefore is the value of $\hat{\theta} \cdot \hat{k}$.



Making this substitution,

$$F = \oiint -\tau_{\theta r} \sin\theta\, dS$$

This is now a scalar quantity we can integrate easily. We note that $dS = r^2 \sin\theta\, d\theta d\phi$ in spherical coordinates, and substituting this in (and applying $r = a$), we arrive at

$$F = \int_0^{2\pi} \int_0^{\pi} -\tau_{\theta r} \sin^2\theta \, a^2 d\theta d\phi$$

When we substituting in $\tau_{\theta r}$ (evaluated at $r = a$), we arrive at

$$F = \int_0^{2\pi} \int_0^{\pi} -\left(-\frac{3}{2}\frac{\eta U \sin\theta}{a}\right) \sin^2\theta \, a^2 d\theta d\phi$$

which simplifies to

$$F = \frac{3}{2}\eta U a \int_0^{2\pi} \int_0^{\pi} \sin^3\theta \, d\theta d\phi$$

This becomes

$$F = 3\eta U a\pi \int_0^{\pi} \sin^3\theta \, d\theta$$

We can split this into

$$F = 3\eta U a\pi \int_0^{\pi} \sin^2\theta \sin\theta \, d\theta$$

By using a trigonometric identity,

$$F = 3\eta U a\pi \int_0^{\pi} (1 - \cos^2\theta) \sin\theta \, d\theta$$

By using the substitution $\xi = \cos\theta$ and $d\xi = -\sin\theta \, d\theta$ we can say that that

$$F = -3\eta U a\pi \int_{\theta=0}^{\theta=\pi} (\xi^2 - 1) d\xi$$

which becomes

$$F = 4\pi\eta U a$$

Assigning the sign yields

$$\vec{F} = 4\pi\eta U a \, \hat{k}$$

3.1.3 FORCE DUE TO VISCOUS NORMAL STRESS

The total force in the r direction can be found by

$$\vec{F} = \oiint f_r \hat{r} \, dS$$

Substituting in the value of f_r yields

$$\vec{F} = \oiint (-P + \tau_{rr})\hat{r}\, dS$$

This can be split up into two parts:

$$\vec{F} = \oiint -P\hat{r}\, dS + \oiint \tau_{rr}\hat{r}\, dS$$

The right integral is the force due to the viscous normal stress and will be determined in this part of the problem. Just to be rigorous, we once again note that it is difficult to calculate the surface integral of a vector and must multiply both sides by $\hat{k}$, the direction of the force. As such,

$$F = \vec{F} \cdot \hat{k} = \oiint -P\hat{r} \cdot \hat{k}\, dS + \oiint \tau_{rr}\hat{r} \cdot \hat{k}\, dS$$

The left integral is the force due to pressure, and the right integral is the force due to the viscous normal stress. I will focus on the force due to the viscous normal stress in this subsection. It can be calculated by

$$\tau_{rr} = 2\eta \frac{\partial u_r}{\partial r}$$

By evaluating this at $r = a$, we arrive at

$$\tau_{rr} = 0$$

Therefore, when we go to calculate the force due to the viscous normal stress, we would find that

$$\oiint \tau_{rr}\hat{r} \cdot \hat{k}\, dS = 0$$

and so there is no contribution from the viscous normal stress.

3.1.4 FORCE DUE TO PRESSURE

We will now calculate the force due to pressure (i.e. the left surface integral in the previous subsection). It was explicitly derived in the previous subsection, but I will drive this point home by recalling that $\vec{f} = -P\hat{n}$. As such, since $\hat{n} = \hat{r}$ and since we must convert the vector to a scalar for integration purposes, we arrive at the same result as in the prior subsection

$$F = \oiint -P\hat{r} \cdot \hat{k}\, dS$$

We can now note that $\hat{r} \cdot \hat{k} = \cos\theta$. Once again, this is not a trivial point, but it can be determined from the previous figure by focusing on the maroon arrows that represent the normal force. We see that the value of the force should align with $\hat{k}$ at $\theta = 0$ and be in the direction of $-\hat{k}$ at $\theta = \pi$. Further, we see that at $\theta = \pi/2$ and $\theta = 3\pi/2$, the normal force is orthogonal to the direction of fluid flow and therefore has no component in the $\hat{k}$ direction. The appropriate trigonometric function for this is $\cos\theta$.

Applying this identity,

$$F = \oiint -P\cos\theta\, dS$$

We recall that $dS = r^2 \sin\theta\, d\theta d\phi$ in spherical coordinates. Substituting this in and making $r = a$, we arrive at

$$F = \int_0^{2\pi} \int_0^{\pi} -P \cos\theta \, a^2 \sin\theta \, d\theta d\phi$$

Substituting in for the pressure

$$F = \int_0^{2\pi} \int_0^{\pi} -\left(P_0 - \frac{3}{2}\frac{\eta U}{a}\cos\theta\right) \cos\theta \, a^2 \sin\theta \, d\theta d\phi$$

This simplifies to

$$F = \int_0^{2\pi} \int_0^{\pi} \frac{3}{2}\eta U a \cos^2\theta \sin\theta \, d\theta d\phi - \int_0^{2\pi} \int_0^{\pi} P_0 a \cos\theta \sin\theta \, d\theta d\phi$$

The right term goes to zero because the integral of an odd function over a symmetric range is zero (or you can calculate it yourself to find out). This means that

$$F = \int_0^{2\pi} \int_0^{\pi} \frac{3}{2}\eta U a \cos^2\theta \sin\theta \, d\theta d\phi$$

This becomes

$$F = 3\pi\eta U a \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta$$

By using the substitution $\xi = \cos\theta$ and $d\xi = -\sin\theta \, d\theta$, we can integrate the expression with ease to yield

$$F = 2\pi\eta U a$$

If we apply the direction now we arrive at

$$\vec{F} = 2\pi\eta U a \, \hat{k}$$

3.1.5 TOTAL FORCE ON THE SPHERE FROM THE FLUID

The total force on the sphere from the fluid is the sum of the forces in the prior three sections. As such, the total force is simply

$$\vec{F} = 6\pi\eta U a \, \hat{k}$$

This is often referred to as Stokes' formula.

4 ONE-DIMENSIONAL FLOW IN VISCOUS FLUIDS

4.1 POISEUILLE FLOW

Let us consider the steady flow of an incompressible fluid through a horizontal cylinder of length L and radius a. The goal is to find the velocity profile, mean velocity, and volumetric flow rate.

In this problem, I will use cylindrical (r, θ, z) components. The z direction will be the direction down the pipe. The value of $r = 0$ will be set to be in the middle of the cylindrical pipe. With this set of definitions, we have that $\vec{u} = u_z$ (i.e. $u_r = u_\theta = 0$). We also postulate that $u_z(r)$.

We start with the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{u} = 0$$

Assuming that ρ is constant, this simply becomes

$$\nabla \cdot \vec{u} = 0$$

which in cylindrical coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(u_\theta) + \frac{\partial}{\partial z}(u_z) = 0$$

This simplifies to

$$\frac{\partial u_z}{\partial z} = 0$$

Now, let us write the Navier-Stokes equation. Ignoring the effect of gravity,

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u}\right) = -\nabla P + \eta \nabla^2 \vec{u}$$

In cylindrical coordinates this becomes,

$$\rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \eta\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}\right)$$

We employ the previous assumptions. Also, we note that the pressure differential can be well-approximated by a linear pressure drop, ΔP, which is conventionally defined to be a positive quantity. As such,

$$\frac{dP}{dz} = \frac{P_2 - P_1}{L} = -\frac{\Delta P}{L}$$

As such, the Navier-Stokes equation can be simplified to the following form

$$0 = \frac{\Delta P}{L} + \eta\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)\right)$$

Rearranging the above expression yields

$$-\frac{\Delta P}{\eta L} r = \frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)$$

Integrating twice yields

$$u_z = -\frac{\Delta P}{\eta L}\frac{r^2}{4} + C_1 \ln(r) + C_2$$

We have to employ boundary conditions now. We know that at $r = 0$, the velocity should be finite. As such, we can immediately say that $C_1 = 0$. Therefore,

$$u_z = -\frac{\Delta P}{\eta L}\frac{r^2}{4} + C_2$$

The other condition is that at $r = a$, the no-slip boundary condition applies and $u_z = 0$. We then have that

$$C_2 = \frac{\Delta P}{\eta L}\frac{a^2}{4}$$

such that

$$u_z = \frac{\Delta P}{4\eta L}(a^2 - r^2)$$

The mean velocity can be calculated by dividing the total volumetric flow rate by the cross-sectional area via

$$\langle u_z \rangle = \frac{\iint u_z \, dA}{\iint dA}$$

This becomes

$$\langle u_z \rangle = \frac{\int_0^{2\pi} \int_0^a u_z \, r dr d\theta}{\int_0^{2\pi} \int_0^a r dr d\theta} = \frac{\frac{\pi a^4 \Delta p}{8\eta L}}{\pi a^2} = \frac{a^2 \Delta P}{8\eta L}$$

The mean velocity in the cross-section is then

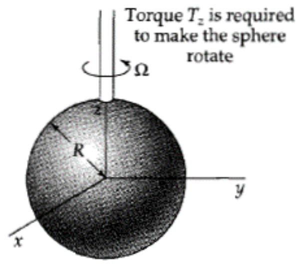
$$\langle u_z \rangle = \frac{a^2 \Delta P}{8\eta L}$$

The total (volumetric) flow rate can be found by multiplying the mean velocity in the cross-section by the cross-sectional area

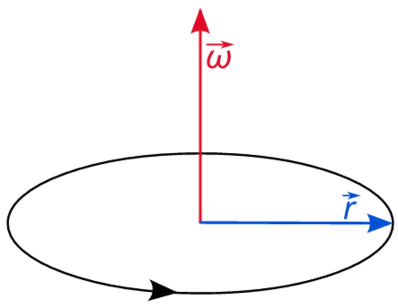
$$Q = \iint \langle u_z \rangle \, dA = \int_0^{2\pi} \int_0^a \langle u_z \rangle \, r dr d\theta = \frac{\pi a^4 \Delta P}{8\eta L}$$

4.2 STOKES FLOW AROUND A SPHERE: TRIAL SOLUTIONS

Consider a sphere rotating very slowly in the $\hat{\phi}$ direction (in spherical coordinates) with an angular velocity $\vec{\Omega}$. An image is shown below. We want to solve for the velocity profile and the torque of the fluid on the sphere.



Before diving into the problem, I will remind you what angular velocity is. Angular velocity has units of inverse time. It has a direction along the axis of rotation (as defined by the right-hand rule). For instance, if the sphere is rotating counterclockwise, the direction would be upward as shown below. Also, the relationship between angular velocity and velocity is the distance from the axis of rotation.



We start with the continuity equation. For constant density, we have the following in spherical coordinates

$$\frac{\partial u_\phi}{\partial \phi} = 0$$

Now, we write the Navier-Stokes equation in the direction of fluid flow, which is ϕ, to get

$$0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(u_\phi \sin\theta)\right)$$

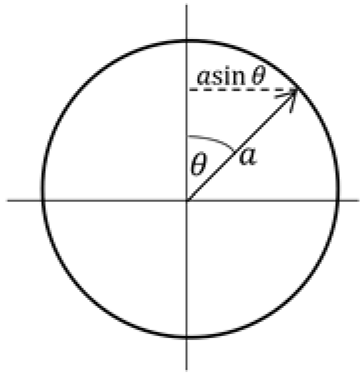
We have the following boundary conditions:

$$r = a, \quad u_\phi = a\Omega \sin\theta$$

$$r \to \infty, \quad u_\phi = 0$$

Let us pause for a second to figure out how the boundary conditions were obtained. On the surface of the sphere, the no-slip boundary condition applies. We are given angular velocity, which has units of 1/s. We want to convert this into the velocity in the ϕ direction. There are two ways we can figure out how to get this. The first is a purely mathematical argument. At the top of the sphere ($\theta = 0$) and the bottom of the sphere ($\theta = \pi$), the value of u_ϕ should be zero because there is no rotation at the vertical poles. Conversely, at $\theta = \pi/2$, the velocity in the ϕ direction should simply be given by $a\Omega$. The sine function is zero at $\theta = 0, \pi$ so we can state that the aforementioned conditions are satisfied if we have $u_\phi = a\Omega \sin\theta$ at $r = a$. The second way is a purely geometrical argument. Consider the following schematic. The velocity in the ϕ

direction is related to the distance from the axis of rotation, not from the center of the sphere. To find this quantity, we see that we need to a factor of $a\sin\theta$ such that $u_\phi = a\Omega\sin\theta$ at $r = a$.



The second way is a purely geometric argument. Consider the following schematic. To find the dostamce between the surface and the axis of rotation, we see that we need to a factor of $a\sin\theta$ such that $u_\phi = a\Omega\sin\theta$ at $r = a$.

By looking at the boundary condition at $r = a$, it is reasonable to assume a solution of the form

$$u_\phi = f(r)\sin\theta$$

When this trial solution is inserted into the Navier-Stokes equation and simplified through the use of the product rule, it results in the following expression

$$0 = \frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) - 2f$$

This is called an equidimensional equation and can be solved with a trial solution of the form

$$f = r^n$$

where the powers of n will be used to generate an expression for f that has constant coefficients raised to the power of n. When substituting in $f = r^n$ into the simplified Navier-Stokes equation we see that $n = -2, 1$. Therefore, the solution takes the form

$$f = C_1 r + \frac{C_2}{r^2}$$

Recall that we said $u_\phi = f(r)\sin\theta$, so

$$u_\phi = \left(C_1 r + \frac{C_2}{r^2}\right)\sin\theta$$

We now employ the boundary conditions to find that $C_1 = 0$ and $C_2 = \Omega a^3$ so that the velocity profile is

$$u_\phi = \frac{\Omega a^3}{r^2}\sin\theta$$

If the pressure field is desired, one can solve the Navier-Stokes equations in the other dimensions (you would find that pressure is constant when this is done). The torque can be determined by computing the stress, multiplying it by the lever arm, and then integrating over the surface of the sphere. Since we have flow in the ϕ direction that is a function of r, we want the stress that is $\tau_{r\phi}$ which is

$$\tau_{r\phi} = \eta r \frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right)$$

in our case once the simplifications are made. You can plug in the velocity distribution and apply $r = a$ (since we want the stress at the surface) to get

$$\tau_{r\phi}\big|_{r=a} = -3\eta\Omega \sin\theta$$

The lever arm is given by $a\sin\theta$, so the torque can be found by

$$K = \oiint \tau_{r\phi}\big|_{r=a}(a\sin\theta)\, dS$$

Once this computation is performed, you arrive at

$$K = -8\pi\eta\Omega a^3$$

Of course, torque is a vector, and it needs a direction. It will be in the direction of the angular velocity. As such,

$$\vec{K} = -8\pi\eta\vec{\Omega}a^3$$

4.3 PLATE SUDDENLY SET IN MOTION: TIME-DEPENDENT FLOW

Consider a semi-infinite body of liquid at a constant density and viscosity that is sitting atop a horizontal plate in the xz plane. The plate is suddenly set into motion at a velocity u_0, causing a fluid velocity profile that changes in both time and in y, the vertical distance from the plate. The goal is to find the velocity profile.

As always, we start with the Navier-Stokes equation

$$\rho\left(\frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) = -\nabla P + \eta\nabla^2\vec{u} + \rho\vec{g}$$

The fluid velocity is $\vec{u} = u_x(y,t)$. Therefore, the above equation simplifies to the following once relevant terms are canceled in the Navier-Stokes equation:

$$\frac{\partial u_x}{\partial t} = \nu\frac{\partial^2 u_x}{\partial y^2}$$

where $\nu \equiv \eta/\rho$. We have the following conditions:

$$t \le 0, \quad u_x = 0$$

$$y = 0, \quad u_x = u_0$$

$$y = \infty, \quad u_x = 0$$

To solve this, we need to first introduce a non-dimensional quantity for the velocity. I will define the non-dimensional velocity as

$$\phi \equiv \frac{u_x}{u_0}$$

The boundary conditions now become the following for $\phi(y,t)$

$$\phi(y,0) = 0, \quad \phi(0,t) = 1, \quad \phi(\infty,t) = 0$$

We know that ϕ should be a quantity that is proportional to y, t, and ν (our independent and dependent variables in the simplified Navier-Stokes equation). We can then say that

$$\phi = \phi(\eta)$$

where

$$\eta \equiv \frac{y}{\sqrt{4\nu t}}$$

It should be apparent that the dimensions of η are indeed unitless. I have included the factor of 4 because I know what the answer is going to be in advance and it simplifies the algebra. This step is not necessary and does not change the validity of the solution. With these expressions, we can rewrite the partial differential equation as

$$\frac{\partial \phi}{\partial t} = \nu \frac{\partial^2 \phi}{\partial y^2}$$

Let's break this down part by part. For the time component we can say that

$$\frac{\partial \phi}{\partial t} = \frac{d\phi}{d\eta}\frac{\partial \eta}{\partial t} = -\frac{1}{2}\frac{\eta}{t}\frac{d\phi}{d\eta}$$

For the y component we can say that

$$\frac{\partial \phi}{\partial y} = \frac{d\phi}{d\eta}\frac{\partial \eta}{\partial y} = \frac{d\phi}{d\eta}\frac{1}{\sqrt{4\nu t}}$$

Therefore,

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{d^2\phi}{dy^2}\frac{1}{4\nu t}$$

This makes the N-S equation become

$$\frac{d^2\phi}{d\eta^2} + 2\eta\frac{d\phi}{d\eta} = 0$$

If I tentatively define

$$\psi \equiv \frac{d\phi}{d\eta}$$

such that

$$\frac{d\psi}{d\eta} + 2\eta\psi = 0$$

we can then rearrange this to

$$\frac{1}{\psi} d\psi = -2\eta \, d\eta$$

and integrate once to get

$$\psi = C_1 \exp(-\eta^2)$$

Transforming this back to our prior set of variables,

$$\frac{d\phi}{d\eta} = C_1 \exp(-\eta^2)$$

And integrating one final time yields

$$\phi = C_1 \int_0^\eta \exp(-\bar{\eta}^2) \, d\bar{\eta} + C_2$$

where I have set $\bar{\eta}$ to be a dummy variable of integration to distinguish it from η in our integral's bounds. The boundary conditions are

$$\eta = 0, \quad \phi = 1$$

$$\eta = \infty, \quad \phi = 0$$

Applying these boundary conditions yields the following after some algebra,

$$\phi = 1 - \frac{\int_0^\eta \exp(-\bar{\eta}^2) \, d\bar{\eta}}{\int_0^\infty \exp(-\bar{\eta}^2) \, d\bar{\eta}} = 1 - \frac{2}{\sqrt{\pi}} \int_0^\eta \exp(-\bar{\eta}^2) \, d\bar{\eta} = 1 - \mathrm{erf}(\eta) = \mathrm{erfc}(\eta)$$

This solution makes use of the error function, denoted erf, but do not let this scare you – it is simply a short-hand way of expressing the otherwise messy integral shown above. The complementary error function, erfc, is simply 1 minus the error function. With this, we can transform ϕ back to our original variables to arrive at

$$u_x = u_0 \, \mathrm{erfc}\left(\frac{y}{\sqrt{4\nu t}}\right)$$

5 VORTICITY

5.1 DEFINITION OF VORTICITY

Vorticity is defined as

$$\vec{w} = \nabla \times \vec{u}$$

The vorticity indicates the local rate of rotation of a fluid element. Generally, it is different from point to point. The surface integral of a curl can be related to the line integral of velocity via Stokes' theorem (see Appendix):

$$\oiint \vec{w} \cdot \hat{n}\, dS = \oint \vec{u} \cdot d\vec{r}$$

The line integral of velocity is also called the circulation of velocity (over a given boundary). To test this formula out, consider a disk in rotation with a radius a. The above expression then lets us say that

$$\vec{w}\pi a^2 = (\vec{\Omega} a) 2\pi a$$

such that

$$\vec{w} = 2\vec{\Omega}$$

To clarify, the left-hand side of the first equation is vorticity multiplied by the surface area of the disk whereas the right-hand side of the first equation is the velocity multiplied by the circumference of the disk.

In the following sections, we will find that

1. Vorticity is generated on solid surfaces due to no-slip boundary condition
2. Vorticity "diffuses" due to viscosity
3. Vorticity is swept downstream due to convection

5.2 CURL OF NAVIER-STOKES

The curl of the Navier-Stokes equation (in dimensionless form) is the following:

$$\frac{\partial \vec{w}}{\partial t} + \vec{u} \cdot \nabla \vec{w} = \vec{w} \cdot \nabla \vec{u} + \frac{1}{\mathrm{Re}} \nabla^2 \vec{w}$$

In a 2D or axisymmetric flow, we have that $\vec{w} \cdot \nabla \vec{u} = 0$, so the Navier-Stokes equation becomes

$$\frac{\partial \vec{w}}{\partial t} + \vec{u} \cdot \nabla \vec{w} = \frac{1}{\mathrm{Re}} \nabla^2 \vec{w}$$

5.3 LOW REYNOLDS NUMBER

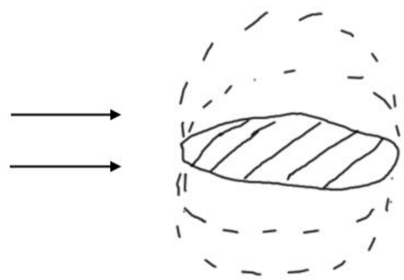
One extreme case we will consider is when the Reynolds number is very small (approaching zero), such as with creeping flow. Recall that this is the same as creeping flow, so the entire left-hand side of the (velocity-pressure form) Navier-Stokes equation drops out. When dealing with the vorticity form, since convection is negligible, nd we can write the Navier-Stokes equation in dimensional variables as follows

$$\frac{\partial \vec{w}}{\partial t} = \frac{\eta}{\rho} \nabla^2 \vec{w}$$

where η/ρ is the kinematic viscosity, often denoted ν. This equation is that of the diffusion equation. Of course, if the vorticity does not change with time we arrive at

$$\nabla^2 \vec{w} = 0$$

This essentially means that for creeping flow with low Reynolds number, and the vorticity looks like the following where each line represents a constant vorticity contour line and the flow comes from the left.



5.4 HIGH REYNOLDS NUMBER

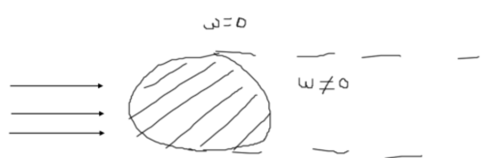
If the Reynolds number is very large (approaching infinity), $\frac{1}{\mathrm{Re}}\nabla^2\vec{w}$ can be ignored, and the Navier-Stokes equation simplifies to

$$\frac{\partial \vec{w}}{\partial t} + \vec{u} \cdot \nabla \vec{w} = 0$$

which is the same as

$$\frac{D\vec{w}}{Dt} = 0$$

This states that the vorticity is conserved (i.e. it remains constant) in each moving fluid element. In a steady flow, the vorticity is then constant along a given streamline. When the Reynolds number is high, we have increasing convection, and the vorticity field around an arbitrary body looks as follow, with $\vec{w} = 0$ outside the boundary layer and wake but $\vec{w} \neq 0$ inside the boundary layer and wake.



While the above form of the Navier-Stokes equation tells us physical information about the vorticity, it is not incredibly useful for gaining information about the velocity profiles since the boundary conditions for vorticity are not straightforward. We would also like to write the velocity-pressure form of the Navier-Stokes equation but in a way that accounts for regions of irrotational flow.

From vector calculus (see Appendix) and under the conditions of no divergence of velocity, we know that the following is true

$$\nabla^2 \vec{u} = -\nabla \times (\nabla \times \vec{u}) = -\nabla \times \vec{w}$$

This is an important quantity to know because then we can say that in irritational flow, where $\vec{w} = 0$, the viscous force $\eta \nabla^2 \vec{u} = 0$ for any value of viscosity (of course, this also holds true if the viscosity is incredibly small) even though the viscous stresses are not necessarily zero. Therefore, outside the boundary layer and wake where there is irrotational flow, we can say that the Navier-Stokes equation simplifies to

$$\rho\left(\frac{\partial\vec{u}}{\partial t}+\vec{u}\cdot\nabla\vec{u}\right)=-\nabla P+\rho\vec{g}$$

In steady flow, the time derivative goes away to yield

$$\rho(\vec{u}\cdot\nabla\vec{u})=-\nabla P+\rho\vec{g}$$

This equation is called Euler's equation. After a bit of vector calculus and algebra which has been omitted here for brevity, we can arrive at

$$\frac{u^2}{2}+\frac{P}{\rho}+gz=\text{constant}$$

throughout the irrotational flow areas. This formula is called Bernoulli's theorem.

In regions with irrotational flow (also known as potential flow), we can define ϕ as the velocity potential such that

$$\vec{u}\equiv\nabla\phi$$

This holds because the curl of velocity is zero. Then, from the continuity equation we can say that

$$\nabla\cdot\vec{u}=0$$

which implies

$$\nabla^2\phi=0$$

5.5 CIRCULATION

Suppose a closed curve made up of fluid particles and moving with a fluid where the viscous force is zero or negligible at all points along it. Consider the circulation of velocity along the curve:

$$\text{circulation}=\oint\vec{u}\cdot d\vec{r}$$

We would like to know how the circulation changes with time, or

$$\frac{d}{dt}\oint\vec{u}\cdot d\vec{r}$$

We can distribute the derivative inside and then note that the derivative of the position vector is velocity

$$\oint\frac{d}{dt}(\vec{u}\cdot d\vec{r})=\frac{d\vec{u}}{dt}\cdot d\vec{r}+\vec{u}\cdot\frac{d}{dt}(d\vec{r})=\frac{d\vec{u}}{dt}\cdot d\vec{r}+\vec{u}\cdot d\vec{u}=\frac{d\vec{u}}{dt}\cdot d\vec{r}+d\left(\frac{1}{2}\vec{u}\cdot\vec{u}\right)$$

Therefore,

$$\frac{d}{dt}\oint\vec{u}\cdot d\vec{r}=\frac{d\vec{u}}{dt}\cdot d\vec{r}+d\left(\frac{1}{2}\vec{u}^2\right)$$

We then note that $\oint df=0$ for any single-value function. Thus, in general, for any closed curve

$$\frac{d}{dt}\oint\vec{u}\cdot d\vec{r}=\oint\frac{d\vec{u}}{dt}\cdot d\vec{r}$$

This says that the time derivative of circulation of velocity over a closed curve is equal to the circulation of acceleration over the same curve. If on the curve, the viscous force $-\nu\nabla\times\vec{w}$ is negligible then

$$\frac{d\vec{u}}{dt} = -\frac{1}{\rho}\nabla P$$

Therefore, plugging this into the rate of change of circulation from above (and if ρ is constant)

$$\frac{d}{dt}\oint \vec{u}\cdot d\vec{r} = -\oint \frac{1}{\rho}\nabla P\cdot d\vec{r} = -\frac{1}{\rho}\oint dP = 0$$

Therefore, if ρ is constant, the circulation does not change. This is known as Kelvin's theorem.

6 BOUNDARY LAYER THEORY

6.1 HIGH REYNOLDS NUMBER FLOW OVER A FLAT PLATE PARALLEL TO FLOW

Recall from transport phenomena that the vorticity diffusion due to viscosity can be thought of as

$$\text{(penetration depth of vorticity diffusion over time } t) \sim \sqrt{\nu t}$$

This relationship will prove quite useful.

Consider a flat plate of very small thickness and a length ℓ. It is placed in a steady uniform stream of fluid (with speed U), with the stream parallel to the length. In the absence of any effects of viscosity, the plate causes no disturbance to the stream and the fluid velocity is uniform. However, real fluids have no-slip boundary conditions that slow down the fluid near the liquid-solid interface. The boundary layer thickness will be small compared to length l provided that $\ell U/\nu \gg 1$. The velocity just outside the boundary layer is effectively unchanged and is therefore equal to U. The pressure outside the boundary layer is also uniform and is approximately uniform throughout the boundary layer as well. We can postulate that the boundary layer thickness would be given by[1]

$$\delta \sim \sqrt{\nu t} \sim \sqrt{\nu x/U}$$

This then says that the further away from the leading edge of the plate you are, the larger the boundary layer thickness, as would be expected. The stress can be estimated as

$$\tau_{\text{wall}} \sim \frac{\eta U}{\delta} \sim \eta U^{\frac{3}{2}} \nu^{-\frac{1}{2}} x^{-\frac{1}{2}} \sim \rho \nu^{\frac{1}{2}} U^{\frac{3}{2}} x^{-\frac{1}{2}}$$

The exact solution is

$$\tau_{\text{wall}} = \eta \left(\frac{\partial u}{\partial y}\right)_{y=0} = 0.33 \rho \nu^{\frac{1}{2}} U^{\frac{3}{2}} x^{-\frac{1}{2}} = 0.33 \eta U \left(\frac{U}{\nu x}\right)^{\frac{1}{2}}$$

The velocity can be found from the stress by integrating, which yields

$$u = 0.33 \nu^{-\frac{1}{2}} U^{\frac{3}{2}} x^{-\frac{1}{2}} y + C$$

To find the constant, we employ the no-slip boundary condition, which yields $C = 0$.

$$u = 0.33 \nu^{-\frac{1}{2}} U^{\frac{3}{2}} x^{-\frac{1}{2}} y$$

The drag force per unit length exerted on the two sides of the plate is given by

$$F_{D,\text{per width}} = 2 \int_0^{\ell} \tau_{\text{wall}}\, dx = 1.33 \rho \nu^{\frac{1}{2}} U^{\frac{3}{2}} L^{\frac{1}{2}}$$

As such, the drag force on a plate of width L, that becomes[2]

$$F_D = 1.33 \rho \nu^{\frac{1}{2}} U^{\frac{3}{2}} L^{\frac{3}{2}}$$

[1] Note that if we have a disk spinning in a fluid, we replace U with Ωx to arrive at $\delta \sim \sqrt{\nu/\Omega}$.

[2] More generally, for a width W, it is $F_D = 1.33 \rho \nu^{\frac{1}{2}} U^{\frac{3}{2}} W L^{\frac{1}{2}}$

If one wants to write the boundary layer equations (the analogous to the Navier-Stokes equation), we note that the velocity gradient in the x direction is significantly smaller than that in the y direction. If we start with the Navier-Stokes equation in the x direction as

$$\rho\left(u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y}\right) = -\frac{\partial P}{\partial x} + \eta\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}\right)$$

we will see that the $\partial^2 u_x / \partial x^2$ term in the Laplacian can be ignored since the velocity gradient in x is small. Note that the $\frac{\partial u_x}{\partial x}$ term in the left-hand side of the equation cannot be dropped because u_y is small and therefore $u_x\, \partial u_x / \partial x$ is not significantly smaller than $u_y\, \partial u_x / \partial y$. Therefore,

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} - \nu \frac{\partial^2 u_x}{\partial y^2} = -\frac{1}{\rho}\frac{\partial P}{\partial x}$$

From Bernoulli's equation, we know that $P + \frac{1}{2}\rho U^2 = \text{constant}$. Taking the x derivative of both sides yields

$$\frac{dP}{dx} = \frac{d\left(\frac{1}{2}\rho U^2 + C\right)}{dx}$$

This becomes

$$\frac{dP}{dx} = -\rho U \frac{dU}{dx}$$

We can substitute this into the boundary layer equation to arrive at

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} - \nu \frac{\partial^2 u_x}{\partial y^2} = U \frac{dU}{dx}$$

If U is constant, then

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = \nu \frac{\partial^2 u_x}{\partial y^2}$$

Naturally, the continuity equation can also be written and is given by

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

While these equations are originally derived for a flat plate, they also apply to flow around a cylinder oriented in the same direction as the plate. In that case, x is the distance from the leading edge of the cylinder whereas y is the distance normal to the surface of the cylinder.

6.2 LOW REYNOLDS NUMBER FLOW OVER A FLAT PLATE (ANY DIRECTION)

For a similar flat plate as the previous scenario but now in low Reynolds flow and oriented in any direction (not necessarily parallel to flow), we know that the left-hand side of the Navier-Stokes equation becomes zero due to the creeping flow approximation. As such,

$$0 = -\nabla P + \eta \nabla^2 \vec{u}$$

We can do dimensional analysis to find the drag force. We expect the force to be a function of L, U, and η. Importantly, it is not a function of ρ since that term drops out at low Reynolds numbers. We then find that

$$F_d \sim \eta u L$$

The difference between a parallel and perpendicular plate is just a numerical factor:

$$\frac{F_{d,\text{parallel}}}{F_{d,\text{perp}}} = \frac{1}{2}$$

6.3 HIGH REYNOLDS NUMBER FLOW OVER A FLAT PLATE PERPENDICULAR TO FLOW

Let us recall from the prior section that the flow around a body at high Reynolds number creates a boundary layer forms in the wake of the object.

The area where vorticity is not zero (inside the layer) is called the vortex sheet. Outside the vortex sheet, the vorticity is zero. This means that the velocity can be written as the gradient of a potential function outside the boundary layer:

$$\vec{u} = \nabla\phi$$

and since divergence of velocity is zero

$$\nabla^2\phi = 0$$

Further, outside the boundary layer we know that the Bernoulli equation applies. A general approach to boundary layer problems is then as follows:

1. Since the boundary layer is thin at Re → ∞, find the velocity profile in the irrotational region by solving $\nabla^2\phi = 0$ outside the body
2. Find the pressure outside the boundary layer by using Bernoulli's equation (it is approximately the same as the pressure inside the boundary layer but Bernoulli's equation does not apply there)
3. Solve the boundary layer equations of motion and find shear stresses as needed

It can be shown that the drag force of a plate perpendicular to high Reynolds number flow is

$$F_d \propto \frac{1}{2}\rho U^2 L^2$$

As such,

$$\frac{F_{d,parallel}}{F_{d,\perp}} \propto \frac{\rho \nu^{\frac{1}{2}} U^{\frac{3}{2}} L^{\frac{3}{2}}}{\rho U^2 L^2} \propto \nu^{\frac{1}{2}} U^{-\frac{1}{2}} L^{-\frac{1}{2}} = \frac{1}{\sqrt{\text{Re}}} \ll 1$$

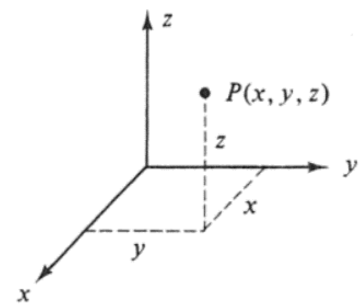
This shows that a parallel plate in high Reynolds number flow has effectively no drag force compared to a plate perpendicular to the follow.

7 APPENDIX: VECTOR CALCULUS

7.1 COORDINATE SYSTEMS

7.1.1 CARTESIAN COORDINATE SYSTEM

The following diagram is a schematic of the Cartesian coordinate system.

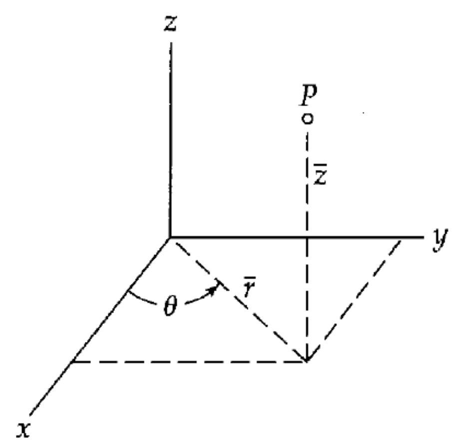


With this definition, the position vector in Cartesian coordinates is

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

7.1.2 CYLINDRICAL COORDINATE SYSTEM

The following diagram is a schematic of the cylindrical coordinate system. Take note that the standard definition is that the sign of the azimuth is considered positive in the counter clockwise direction.



With this definition, the position vector in cylindrical coordinates is

$$\vec{r} = r\hat{r} + z\hat{z}$$

To convert from cylindrical coordinates to Cartesian coordinates,

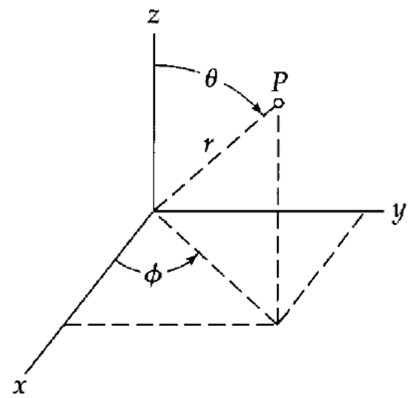
$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

7.1.3 SPHERICAL COORDINATE SYSTEM

The following diagram is a schematic of the spherical coordinate system. Note that many mathematics textbooks use a slightly different convention by swapping the definitions of θ and ϕ. Take note that the standard definition is that the sign of the azimuth is considered positive in the counter clockwise direction and that the inclination angle is the angle between the zenith direction and a given point.



With this definition, the position vector in spherical coordinates is

$$\vec{r} = r\hat{r}$$

To convert from spherical coordinates to Cartesian coordinates,

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

7.1.4 SURFACE DIFFERENTIALS

The surface differentials, dS, in each of the three major coordinate systems are as follows.

Coordinate system	Surface differential, dS
Cartesian (top, $\hat{n} = \hat{z}$)	$dx\ dy$
Cartesian (side, $\hat{n} = \hat{y}$)	$dx\ dz$
Cartesian (side, $\hat{n} = \hat{x}$)	$dy\ dz$
Cylindrical (top, $\hat{n} = \hat{z}$)	$r\ dr\ d\theta$
Cylindrical (side, $\hat{n} = \hat{r}$)	$r\ d\theta\ dz$
Spherical ($\hat{n} = \hat{r}$)	$r^2\sin\theta\ \ d\theta\ d\phi$

7.1.5 VOLUME DIFFERENTIALS

The volume differentials, dV, in each of the three major coordinate systems are as follows.

Coordinate system	Volume differential, dV
Cartesian	$dx\ dy\ dz$
Cylindrical	$r\ dr\ d\theta\ dz$
Spherical	$r^2\sin\theta\ \ dr\ d\theta\ d\phi$

7.2 Mathematical Operations

7.2.1 Magnitude

The magnitude of a vector is its length and can be computed as the following.

In the Cartesian coordinate system

$$|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

In the cylindrical coordinate system

$$|\vec{v}| = \sqrt{v_r^2 + v_z^2}$$

In the spherical coordinate system

$$|\vec{v}| = v_r$$

7.2.2 Dot Product

The dot product of two vectors is

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i v_i$$

The dot product of a tensor with a vector, such as $\vec{f} = \bar{\bar{\sigma}} \cdot \vec{n}$ is what one would expect from matrix algebra:

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} \sigma_{ii} & \sigma_{ij} & \sigma_{ik} \\ \sigma_{ji} & \sigma_{jj} & \sigma_{jk} \\ \sigma_{ki} & \sigma_{kj} & \sigma_{kk} \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

7.2.3 Cross Product

In matrix notation, the cross product is

$$\vec{u} \times \vec{v} = \det \begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_i & u_j & u_i \\ v_i & v_j & v_j \end{pmatrix} = (u_j v_k - u_k v_j)\hat{\imath} + (u_k v_i - u_i v_k)\hat{\jmath} + (u_i v_j - u_j v_i)\hat{k}$$

where i, j, and k represent the three coordinates in the given coordinate system.

7.3 Operators

7.3.1 Gradient

The gradient is a mathematical operator that acts on a scalar function and is written as $\mathrm{grad}(f)$ or ∇f. The result is always a vector. It is essentially the derivative applied to functions of several variables.

In Cartesian coordinates, the gradient is

$$\mathrm{grad}(f) = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

In cylindrical coordinates, the gradient is

$$\text{grad}(f) = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{\partial f}{\partial z}\hat{z}$$

In spherical coordinates, the gradient is

$$\text{grad}(f) = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}$$

7.3.2 DIVERGENCE

The divergence is a mathematical operator that acts on a vector function and is written as $\text{div}(\vec{v})$ or $\nabla \cdot \vec{v}$. The result is always a scalar. The divergence represents the flux emanating from any point of the given vector function (essentially, a rate of loss of a specific quantity).

In Cartesian coordinates, the divergence is

$$\text{div}(\vec{v}) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

In cylindrical coordinates, the divergence is

$$\text{div}(\vec{v}) = \frac{1}{r}\frac{\partial}{\partial r}\left(r v_r\right) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

In spherical coordinates, the divergence is

$$\text{div}(\vec{v}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(v_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

7.3.3 CURL

The curl is a mathematical operator that acts on a vector function and is written as $\text{curl}(\vec{v})$ or $\nabla\times\vec{v}$. The result is always a vector. The curl represents the infinitesimal rotation of a vector function.

In Cartesian coordinates, the curl is

$$\text{curl}(\vec{v}) = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{y} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{z}$$

In cylindrical coordinates, the curl is

$$\text{curl}(\vec{v}) = \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)\hat{r} + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial\left(r v_\theta\right)}{\partial r} - \frac{\partial v_r}{\partial \theta}\right)$$

In spherical coordinates, the curl is

$$\text{curl}(\vec{v}) = \frac{1}{r\sin\theta}\left(\frac{\partial\left(v_\phi \sin\theta\right)}{\partial \theta} - \frac{\partial v_\theta}{\partial \phi}\right)\hat{r} + \left(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{1}{r}\frac{\partial\left(r v_\theta\right)}{\partial r}\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial\left(r v_\theta\right)}{\partial r} - \frac{\partial v_r}{\partial \theta}\right)\hat{\phi}$$

7.3.4 LAPLACIAN

The Laplacian is a mathematical operator that acts on a scalar function and is written as $\nabla^2 f$. The result is always a scalar. It represents the divergence of the gradient of a scalar function.

In Cartesian coordinates, the Laplacian is

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

In cylindrical coordinates, the Laplacian is

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

In spherical coordinates, the Laplacian is

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

7.4 COMMON IDENTITIES OF SECOND DERIVATIVES

The following identities are useful when dealing with second derivative terms. For a scalar field f,

$$\mathrm{div}\big(\mathrm{grad}(f)\big) = \nabla^2 f$$

$$\mathrm{curl}\big(\mathrm{grad}(f)\big) = 0$$

Further, for a vector field $\vec{f}$,

$$\mathrm{div}\left(\mathrm{curl}(\vec{f})\right) = 0$$

$$\nabla^2 \vec{f} = \mathrm{grad}\left(\mathrm{div}(\vec{f})\right) - \mathrm{curl}\left(\mathrm{curl}(\vec{f})\right)$$

In the special case of divergenceless velocity (from the continuity equation), we can then make the simplification that $\nabla^2\vec{u} = -\mathrm{curl}\big(\mathrm{curl}(\vec{u})\big) = -\mathrm{curl}(\vec{w})$.

7.5 SURFACE INTEGRATION

7.5.1 THE SURFACE INTEGRAL

The surface integral is a generalization of multiple integrals to integration over surfaces. It is the two-dimensional extension of the one-dimensional line integral. The notation of the surface integral is not agreed upon. Some texts using a double integral with an S beneath to indicate a surface integral, whereas other texts use the symbol for a line integral – an integral with a circle around the center – to represent surface integrals as well. Some other texts using a double integral with a circle around it. They all mean the same thing.

The surface integral of a scalar field is written and computed as

$$F = \oiint f\, dS$$

The surface integral of a vector field cannot be as easily computed. If one wants to compute the surface integral of, say, the force (which is a vector), one needs to convert it first to a scalar and then apply the direction at the end of the computation. As such, the general method of doing the surface integral of a vector is to say

$$F = \oiint \vec{f} \cdot \hat{k}\, dS$$

where $\hat{k}$ is in the same direction as F is anticipated to be in. In the special case of $\hat{k} = \hat{n}$, this surface integral is called the flux

$$\text{Flux} = \oiint \vec{f} \cdot \hat{n}\, dS$$

To make the computation of surface integrals easier, common systems and their corresponding dS equivalents are included in section 1.1.4. You can then simply substitute in for the surface element dS in the integral to convert it to a standard double integral and then apply the appropriate bounds. For surface integrals of vector fields, be sure to substitute in the appropriate normal vector for the coordinate system.

Note that if the vector field is given in terms of $\vec{f}(x, y, z) = \alpha\hat{x} + \beta\hat{y} + \gamma\hat{z}$ but you are setting up the surface integral for spherical coordinates (e.g. flux along the surface of a sphere), the normal vector is $\hat{n} = \hat{r}$ but it is not apparent how to calculate the dot product of something with a Cartesian unit vector and spherical unit vector. To resolve this difference in coordinate systems, you will need to convert the normal vector into Cartesian coordinates. This can be done by recognizing that $\hat{r} = \vec{r}/|\vec{r}| = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$, and then the dot product can be appropriately taken.

7.5.2 DIVERGENCE THEOREM

The divergence theorem can convert a surface integral into a volume integral when applied to a vector field via

$$\oiint \vec{f} \cdot \hat{n}\, dS = \iiint \operatorname{div}(\vec{f})\, dV$$

The volume integral can be computed by substituting in the appropriate volume element dV and including the appropriate bounds.

7.6 STOKES' THEOREM

Stokes' theorem states can convert a surface integral into a line integral when applied to the curl of a vector field via

$$\oiint (\nabla \times \vec{f}) \cdot \hat{n}\, dS = \oint \vec{f} \cdot d\vec{r}$$

8 APPENDIX: PRACTICAL PROBLEM SOLVING METHODS

8.1 DERIVING EXPRESSIONS FOR VELOCITY, PRESSURE, AND STRESS

With these tools at our disposal, we can solve many types of fluid mechanics problem. The general approach has been outlined below. They will then be used in the following examples. Tabulated expressions for the Navier-Stokes equation and Newton's Law of Viscosity are included in the Appendix.

1. Choose an appropriate coordinate system
2. Determine the direction of flow in this coordinate system (I refer to this as the j direction)
3. Use the continuity equation to provide further simplifications to the system
4. Use physical details from the problem statement and the result of the continuity equation to determine which direction the velocity is a function of (I refer to this as the i direction)
5. For the velocity distribution, solve the Navier-Stokes equation in the direction of fluid flow (the j direction)
6. For the pressure distribution, solve the Navier-Stokes equation in the direction that the pressure is a function of
7. For the stress, τ_{ij}, substitute the velocity distribution into Newton's Law of Viscosity

8.2 COMMON BOUNDARY CONDITIONS

The following are some of the most common boundary conditions (BC's) used in fluid mechanics and help in determining the constants of integration when the Navier-Stokes equation is solved.

- At a solid-liquid interface, the fluid velocity equals the velocity with which the solid surface is moving (in the common case that the solid interface is stationary, then the fluid velocity is zero at the interface). This is called the no-slip boundary condition
- The inlet or outlet boundary conditions may be explicitly specified
- If there is creeping flow around an object, consider the conditions infinitely far out
- If the surface of a fluid is exposed to the atmosphere, the pressure at the surface is therefore P_{atm}
- At a liquid-gas interface that is oriented in a direction x, the stresses τ_{xy} and τ_{xz} are approximately zero, assuming the gas-side velocity gradient is not sufficiently large
- Check for unphysical terms. For instance, if an equation has a $C\ln(x)$ term in it, then if $x = 0$ is physically allowed then $C = 0$ in order to make the equation physically realizable.

8.3 USING NEWTON'S LAW OF VISCOSITY

As derived previously, Newton's Law of Viscosity is

$$\bar{\bar{\tau}} = \mu\left(\nabla\vec{u} + \nabla\vec{u}^{\mathrm{T}}\right)$$

This equation is written in its most general form and is a bit cumbersome to use in this way. I will show this cumbersome way first and then explain how to use it in a practical way.

<u>Rigorous Way</u>:

You must write out the full expression for the stress vector as defined above. For Cartesian coordinates, this would be

$$\bar{\bar{\tau}} = \mu \left(\begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{pmatrix} + \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_y}{\partial x} & \frac{\partial u_z}{\partial x} \\ \frac{\partial u_x}{\partial y} & \frac{\partial u_y}{\partial y} & \frac{\partial u_z}{\partial y} \\ \frac{\partial u_x}{\partial z} & \frac{\partial u_y}{\partial z} & \frac{\partial u_z}{\partial z} \end{pmatrix} \right)$$

Then, based on the problem, cancel relevant terms that go to zero and you have your expression for the stress.

Practical Way:

1. Determine what direction the velocity is a function of (I refer to this as the i direction)
2. Determine the direction of flow in the coordinate system of choice (I refer to this as the j direction)
3. The stress tensor is then written as τ_{ij} and represents the stress on the positive i face acting in the positive j direction[3]
4. The expression of τ_{ij} can then be more simply expressed as $\tau_{ij} = \mu(\nabla_i u_j + \nabla_j u_i)$. Here, I have introduced my own short-hand notation. The operator ∇_i represents the gradient operator in the i direction and u_j represents the velocity in the j direction. Of course, if there is more than one i and/or j values (e.g. if the fluid velocity is in greater than one dimension) you will need more than one expression for τ_{ij}

8.4 CALCULATING MEAN VELOCITY AND FLOW RATE

To calculate the mean velocity through a given area, simply divide the total volumetric flow rate by the cross-sectional area:

$$\langle u \rangle = \frac{\iint u \, dA}{\iint dA}$$

using the appropriate dA elements for the given coordinate system.

To calculate the volumetric flow rate through a cross-section once the mean velocity is known, this can typically be found by multiplying the mean velocity in the cross-section by the cross-sectional area. More generally speaking, the volumetric flow rate can be found by

$$Q = \iint \langle u \rangle \, dA$$

To find the mass flow rate, simply multiply the volumetric flow rate by density.

[3] Note that many textbooks, most notably BSL, define the stress tensor differently with a negative sign in the front.

9 APPENDIX: TABULATED EXPRESSIONS

9.1 EXPRESSIONS FOR NEWTON'S LAW OF VISCOSITY

Recall that Newton's Law of Viscosity is

$$\bar{\bar{\tau}} = \mu\left(\nabla\vec{u} + \nabla\vec{u}^{\mathrm{T}}\right)$$

and that this can be rewritten as

$$\tau_{ij} = \mu\left(\nabla_i u_j + \nabla_j u_i\right)$$

9.1.1 CARTESIAN COORDINATES

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right)$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$$

9.1.2 CYLINDRICAL COORDINATES

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left(r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)$$

$$\tau_{\theta z} = \tau_{z\theta} = \mu\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right)$$

$$\tau_{zr} = \tau_{rz} = \mu\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)$$

9.1.3 SPHERICAL COORDINATES

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left(r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} = \mu\left(\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{u_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi}\right)$$

$$\tau_{\phi r} = \tau_{r\phi} = \mu\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right)\right)$$

9.2 EXPRESSIONS FOR THE CONTINUITY EQUATION

Recall that the continuity equation states

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho\vec{u} = 0$$

9.2.1 CARTESIAN COORDINATES

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) + \frac{\partial}{\partial y}(\rho u_y) + \frac{\partial}{\partial z}(\rho u_z) = 0$$

9.2.2 CYLINDRICAL COORDINATES

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r u_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho u_\theta)+\frac{\partial}{\partial z}(\rho u_z)=0$$

9.2.3 SPHERICAL COORDINATES

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 u_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\rho u_\theta \sin\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(\rho u_\phi)=0$$

9.3 EXPRESSIONS FOR THE NAVIER-STOKES EQUATION

The Navier-Stokes equation for an incompressible fluid is

$$\rho\left(\frac{\partial \vec{u}}{\partial t}+\vec{u}\cdot\nabla\vec{u}\right)=-\nabla P+\mu\nabla^2\vec{u}+\rho\vec{g}$$

9.3.1 CARTESIAN COORDINATES

$$\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left[\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right]+\rho g_x$$

$$\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\left[\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right]+\rho g_y$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left[\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right]+\rho g_z$$

9.3.2 CYLINDRICAL COORDINATES

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+v_z\frac{\partial v_r}{\partial z}-\frac{v_\theta^2}{r}\right)=-\frac{\partial p}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}+\frac{\partial^2 v_r}{\partial z^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right]+\rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+v_z\frac{\partial v_\theta}{\partial z}+\frac{v_r v_\theta}{r}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{\partial^2 v_\theta}{\partial z^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right]+\rho g_\theta$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right]+\rho g_z$$

9.3.3 SPHERICAL COORDINATES

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi}-\frac{v_\theta^2+v_\phi^2}{r}\right)=-\frac{\partial p}{\partial r}$$
$$+\mu\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_r}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial \phi^2}\right]+\rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}+\frac{v_r v_\theta-v_\phi^2\cot\theta}{r}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}$$
$$+\mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(v_\theta\sin\theta)\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial \phi^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}-\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right]+\rho g_\theta$$

$$\rho\left(\frac{\partial v_\phi}{\partial t}+v_r\frac{\partial v_\phi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}+\frac{v_\phi v_r+v_\theta v_\phi\cot\theta}{r}\right)=-\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}$$
$$+\mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(v_\phi\sin\theta)\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial \phi^2}+\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi}+\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right]+\rho g_\phi$$