Transport Phenomena I

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Contents

1 Dimensional Analysis and Scale-Up

1.1 Procedure

1. To solve a problem using dimensional analysis, write down all relevant variables, their corresponding units, and the fundamental dimensions (e.g. length, time, mass, etc.)

   (a) Some important reminders: a newton (N) is equivalent to kg m s^{-2}, a joule (J) is equivalent to kg m^2 s^{-2}, and a pascal (Pa) is equivalent to kg m^{-1} s^{-2}

   (b) $\mu = \left[ML^{-1}t^{-1}\right]$, $P = \left[ML^{-1}t^{-2}\right]$, $F = \left[MLt^{-2}\right]$, Power $= \left[ML^2t^{-3}\right]$

2. The number of dimensionless groups that will be obtained is equal to the number of variables minus the number of unique fundamental dimensions

3. For each fundamental dimension, choose the simplest reference variable

   (a) No two reference variables can have the same fundamental dimensions

4. Solve for each fundamental dimension using the assigned reference variables

5. Solve for the remaining variables using the previously defined dimensions

6. The dimensionless groups can then be computed by manipulating the algebraic equation created in Step 5

7. If scaling is desired, one can manipulate the constant dimensionless equations

1.2 Example

Consider a fan with a diameter D, rotational speed ω, fluid density ρ, power P, and volumetric flow rate of Q. To solve for the dimensionless groups that can be used for multiplicative scaling, we implement the steps from **Section 1.1**:

Variable	Unit	Fundamental Dimension
D	m	L
ω	rad/s	t^{-1}
ρ	kg/m^3	$m \cdot L^{-3}$
Q	m^3/s	$L^3 \cdot t^{-1}$
P	W	$m \cdot L^2 \cdot t^{-3}$

1. Create a table of the variables, as shown above

2. The number of dimensionless groups can be found by: 5 variables - 3 fundamental dimensions = 2 dimensionless groups

3. The reference variables will be chosen as D, ω, and ρ for length, time, and mass, respectively

4. Each fundamental dimension can be represented by $L = [D]$, $m = \left[\rho D^3\right]$, and $t = \left[\omega^{-1}\right]$

5. The remaining variables can be solved by $[Q] = \left[D^3\omega\right] = L^3 \cdot t^{-1}$ and $[P] = \left[\rho P^3 D^5 \omega^3\right] = m \cdot L^2 \cdot t^{-3}$

6. The two dimensionless groups can therefore be found as $N_1 = QD^{-3}\omega^{-1} \equiv Q^{-1}D^3\omega$ and $N_2 = P\rho^{-1}D^{-5}\omega^{-3} \equiv P^{-1}\rho D^5\omega^3$

2 Introduction to Fluid Mechanics

2.1 Definitions and Fundamental Equations

- Newtonian fluids exhibit constant viscosity but virtually no elasticity whereas a non-Newtonian fluid does not have a constant viscosity and/or has significant elasticity
- Pressure is equal to a force per unit area but is involved with the compression of a fluid, as is typically seen in hydrostatic situations
  - Pressure is independent of the orientation of the area associated with it
  - Additionally, $F = P\,dA$
- For a velocity, v, the volumetric flow, Q, through a plane must be,

$$Q = \int v\,dA \rightarrow vA$$

- A mass flow rate can be defined as

$$\dot{m} = \int \rho v\,dA \rightarrow \rho v A = \rho Q$$

- Similarly, a mass of a vertical column of liquid with height z can be found as

$$m = \rho V = \rho A z$$

- Specific gravity with a water reference is defined as

$$s = \frac{\rho_i}{\rho_{H2O}}$$

- The Reynolds Number is defined as the following where L is the traveled length of fluid or diameter for a pipe system

$$\mathrm{Re} = \frac{\rho v L}{\mu}$$

2.2 Hydrostatics

2.2.1 Pressure Changes with Elevation

- For a hydrostatic situation, it is important to note that $\sum F = ma = 0$
- Since pressure changes with elevation,

$$\frac{dP}{dz} = -\rho g$$

  - At constant ρ and g, the above equation becomes the following when integrated from 0 to z

$$P = \rho g z + P_{\text{surface}}$$

- The potential, Φ, of a fluid is defined as

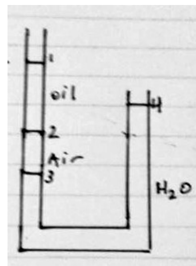
$$\Phi = P + \rho g z = \text{constant}$$

  - This is only true for static liquids with free motion (no barriers) and where z is in the opposite direction of g
- For a multiple fluid U-tube system, the force on the magnitude of the force on the left side must equal the magnitude of the force on the right side

- Therefore, for U-tube with the same area on both sides, the pressure on the left column must equal the pressure on the right column
- Another way to solve this problem is to realize that Φ at the top of a liquid is the same as Φ at the bottom of the same liquid
- The density of air is very small, so ρ_{air} can be assumed to be approximately zero if needed

2.2.2 U-Tube Example

Find ρ_{oil} in the diagram below if $z_{1\to 2} = 2.5\,\text{ft}$, $z_{1\to 3} = 3\,\text{ft}$, $z_{1\to\text{bottom}} = 4\,\text{ft}$, and $z_{4\to\text{bottom}} = 3\,\text{ft}$:



1. Since $\sum F = 0$ for this system and the area is the same on both sides of the U-tube, $P_1 = P_4$
2. Equating the two sides yields

$$P_1 + \rho_{\text{oil}} g\,(2.5\,\text{ft}) + \rho_{\text{air}} g\,(3\,\text{ft} - 2.5\,\text{ft}) + \rho_{\text{water}} g\,(4\,\text{ft} - 3\,\text{ft}) = \rho_{\text{water}} g\,(3\,\text{ft}) + P_4$$

   (a) Note that the depths here are not depths from the surface but, rather, the vertical height of each individual fluid

3. Since $P_1 = P_4$ and $s = \dfrac{\rho_{\text{species}}}{\rho_{\text{water}}}$,

$$s_{\text{oil}} g\,(2.5\,\text{ft}) + s_{\text{air}} g\,(0.5\,\text{ft}) + s_{\text{water}} g\,(1\,\text{ft}) = s_{\text{water}} g\,(3\,\text{ft})$$

4. Canceling the g terms, substituting $s_{\text{water}} \equiv 1$, and assuming $s_{\text{air}} \approx 0$,

$$s_{\text{oil}} = \frac{3\,\text{ft} - 1\,\text{ft}}{2.5\,\text{ft}} = 0.8$$

2.2.3 Force on a Dam

- The force on a dam can be given by the following equation where the c subscript indicates the centroid

$$F = \rho g h_c A = P_c A$$

- For a rectangle of depth D,

$$h_c = \frac{1}{2} D$$

- For a vertical circle of diameter D or radius r,

$$h_c = \frac{1}{2} D = r$$

- For a vertical triangle with one edge coincident with the surface of the liquid and a depth D,

$$h_c = \frac{1}{3} D$$

- The centroid height can be used on any shaped surface. What matters is the shape of the projection of the surface. For instance, a curved dam will have a rectangular projection, and h_c for a rectangle can be used

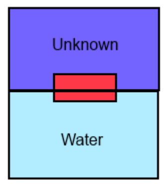
2.2.4 Archimedes' Law

- An object submerged in water will experience an upward buoyant force (has the same magnitude of the weight of the displaced fluid), and at equilibrium it will be equal to the weight of the object downward
- For a submerged object of volume V_o in a fluid of density ρ_f,

$$F_{\text{buoyant}} = \rho_f g V_o = \rho g z A$$

2.2.5 Buoyancy Example

Consider the system below, which is a system of two immiscible fluids, one of which is water (w) and the other is unknown (u). At the interface is a cylinder, which is one-third submerged in the water layer. If the specific gravity of the cylinder is 0.9, find the specific gravity of the unknown:



1. The buoyant force is due to both fluids, so $F_b = \rho_w g A_o \left(\frac{1}{3}\right) + \rho_u g A_o \left(\frac{2}{3}\right)$
2. The weight of the object is $F_o = m_o g = \rho_o A_o \left(\frac{1}{3} + \frac{2}{3}\right)$
3. Equating Step 1 and Step 2 yields, dividing by ρ_w to create specific gravity yields, and canceling out A_o and g yields $\left(\frac{1}{3}\right) s_w + \left(\frac{2}{3}\right) s_u = s_o$
4. The problem statement said that $s_o = 0.9$ and $s_w \equiv 1$, so solving for s_u yields $s_u = 0.85$

3 Shear Stress and the Shell Momentum Balance

3.1 Types of Stress

- Newton's Law of Viscosity states that the viscous stress is given by the following (in Cartesian coordinates)

$$\tau_{ij} = -\mu \left(\frac{\partial v_j}{\partial i} + \frac{\partial v_i}{\partial j}\right)$$

- A tensor of τ_{ij} simply indicates the stress on the positive i face acting in the negative j direction
- To visualize this easier,

$$\tau_{xy} = \tau_{yx} = -\mu \left[\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right]$$

$$\tau_{yz} = \tau_{zy} = -\mu \left[\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right]$$

$$\tau_{zx} = \tau_{xz} = -\mu \left[\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right]$$

3.2 Shell Momentum Balance

3.2.1 Procedure

1. Choose a coordinate system

2. Find the direction of fluid flow. This will be the direction that the momentum balance will be performed on as well as j in τ_{ij}. The velocity in the j direction will be a function of i. This i direction will be the dimension of the shell that will approach zero

3. Find what the pressure in the j direction is a function of (the direction in which $\rho g h$ plays a role)

4. A momentum balance can be written as

$$\sum_{in}(\dot{m}v) - \sum_{out}(\dot{m}v) + \sum_{sys} F = 0$$

   - Frequently, this is simplified to a force balance of

$$\sum_{sys} F = 0$$

   - The relevant forces are usually $F_{\text{gravity}} = \rho g V$, $F_{\text{stress}} = A\tau_{ij}$, and $F_{\text{pressure}} = PA$
     - F_{gravity} is positive if in the same direction of the fluid flow and negative otherwise

5. Divide out constants and let the thickness of the fluid shell approach zero

6. Use the definition of the derivative

$$f'(x_0) \equiv \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

7. Integrate the equation to get an expression for τ_{ij}

   (a) Find the constant of integration by using the boundary condition

8. Insert Newton's law of viscosity and obtain a differential equation for the velocity

9. Integrate this equation to get the velocity distribution and find the constant of integration by using the boundary condition

   (a) To find the average value of the velocity,

$$\langle v_i \rangle = \frac{\iint_D v_i dA}{\iint_D dA}$$

3.2.2 Boundary Conditions

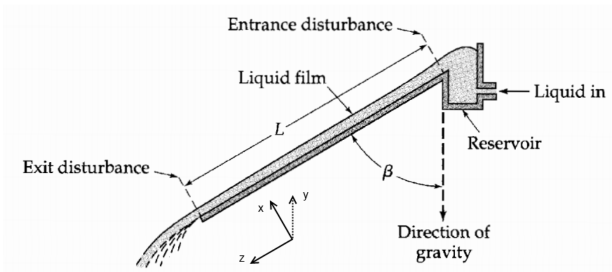
The following boundary conditions are used when finding the constant(s) of integration. Note that at this point, we are no longer dealing with an infinitesimal shell but, rather, the system as a whole

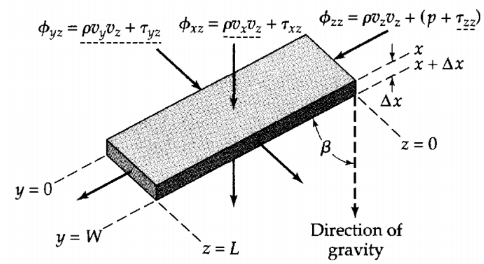
- At solid-liquid interfaces, the fluid velocity equals the velocity with which the solid surface is moving
- At a liquid-liquid interfacial plane of constant i (assuming a coordinate system of i, j, and k), v_j and v_k are constant across the i direction as well as any stress along this plane
- At a liquid-gas interfacial plane of constant i (assuming a coordinate system of i, j, and k), τ_{ij} and τ_{ik} (and subsequently τ_{ji} and τ_{ki}) are zero if the gas-side velocity gradient is not large

- If there is creeping flow around an object, analyze the conditions infinitely far out (e.g. $r \to \infty$ for creeping flow around a sphere)
- It is also important to check for any unphysical terms. For instance, if it is possible for x to equal zero, and the equation has a $C \ln(x)$ term in it, then $C = 0$ since $\ln(0)$ is not possible and thus the term should not even exist

3.3 Flow of a Falling Film

Consider the following system where the z axis will be aligned with the downward sloping plane and its corresponding "shell" shown on the right (graphic from BSL):



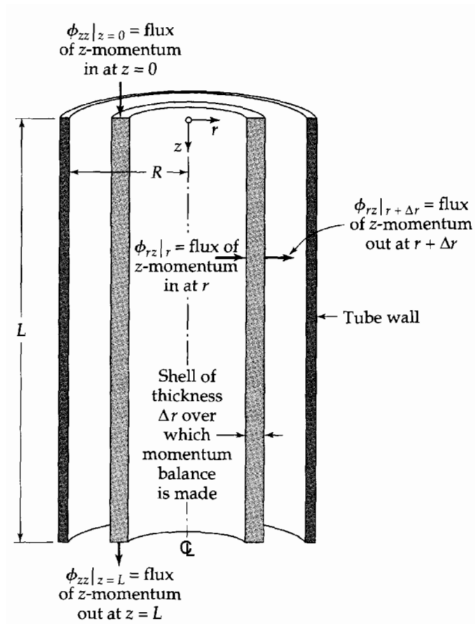


1. Cartesian coordinates will be chosen. The direction of the flow is in the z direction, so this will be the direction of the momentum balance. $v_z(x)$, so the relevant stress tensor is τ_{xz}, and Δx will be the differential element
2. $P(x)$, so it is not included in the z momentum balance
3. Set up the shell balance as
$$LW\left(\tau_{xz}\big|_x - \tau_{xz}\big|_{x+\Delta x}\right) + \rho \Delta x L W g \cos\beta = 0$$
4. Dividing by $LW\Delta x$ and then letting the limit of Δx approach zero yields[1]
$$\lim_{\Delta x \to 0}\left(\frac{\tau_{xz}\big|_{x+\Delta x} - \tau_{xz}\big|_x}{\Delta x}\right) = \rho g \cos\beta$$
5. Note that the first term is the definition of the derivative of τ_{xz}, so
$$\frac{d\tau_{xz}}{dx} = \rho g \cos\beta$$
6. Integrating this equation yields $\tau_{xz} = \rho g x \cos\beta + C_1$, and $C_1 = 0$ since $\tau_{xz} = 0$ at $x = 0$ (liquid-gas interface)
$$\tau_{xz} = \rho g x \cos\beta$$
7. Inserting Newton's law of viscosity of $\tau_{xz} = -\mu \dfrac{dv_z}{dx}$ yields $\dfrac{dv_z}{dx} = -\left(\dfrac{\rho g \cos\beta}{\mu}\right) x$ for the velocity distribution
8. Integrating the velocity profile yields $v_z = -\left(\dfrac{\rho g \cos\beta}{2\mu}\right) x^2 + C_2$, and $C_2 = \left(\dfrac{\rho g \cos\beta}{2\mu}\right)\delta^2$ since $v_z = 0$ at $x = \delta$ if δ is the depth in the x direction of the fluid film
$$v_z = \frac{\rho g \delta^2 \cos\beta}{2\mu}\left[1 - \left(\frac{x}{\delta}\right)^2\right]$$

[1]Note the rearrangement of terms required so that the definition of the derivative has $\tau_{xz}\big|_{x+\Delta x} - \tau_{xz}\big|_x$ and not $\tau_{xz}\big|_x - \tau_{xz}\big|_{x+\Delta x}$

3.4 Flow Through a Circular Tube

Consider the following system (graphic from BSL)



1. This problem is best done using cylindrical coordinates. The fluid is moving in the z direction, $v_z = v_z(r)$, and thus the j tensor subscript is equal to z and i is r. Therefore, the only relevant stress is τ_{rz}. Also, $P(z)$. A z momentum balance will be performed with a differential in the r direction. The force of gravity is in the same direction of fluid flow, so it will be positive

2. Setting up the shell balance yields[2]

$$(2\pi r L \tau_{rz})\big|_r - (2\pi r L \tau_{rz})\big|_{r+\Delta r} + 2\pi r \Delta r\,(P_0 - P_L) + 2\pi r \Delta r L \rho g = 0$$

3. Dividing by $2\pi L \Delta r$, letting the thickness of the shell approach zero, and utilizing the definition of the derivative yields

$$\frac{d\,(r\tau_{rz})}{dr} = \left(\frac{P_0 - P_L}{L} + \rho g\right) r$$

4. While it's algebraically equal to the above equation, a substitution of modified pressure[3] can generalize the equation as

$$\frac{d\,(r\tau_{rz})}{dr} = \left(\frac{\mathscr{P}_0 - \mathscr{P}_L}{L}\right) r$$

5. Integrating the above equation yields $\tau_{rz} = \dfrac{r\,(\mathscr{P}_0 - \mathscr{P}_L)}{2L} + \dfrac{C_1}{r}$, and $C_1 = 0$ since $\tau_{rz} = \infty$ at $r = 0$, which is impossible, so

$$\tau_{rz} = \left(\frac{\mathscr{P}_0 - \mathscr{P}_L}{2L}\right) r$$

6. Using Newton's law of viscosity of $\tau_{rz} = -\mu\left(\dfrac{dv_z}{dr}\right)$ yields

$$\frac{dv_z}{dr} = -\left(\frac{\mathscr{P}_0 - \mathscr{P}_L}{2\mu L}\right) r$$

[2]Note that you cannot do $2\pi r L\left(\tau_{rz}\big|_r - \tau_{rz}\big|_{r+\Delta r}\right)$

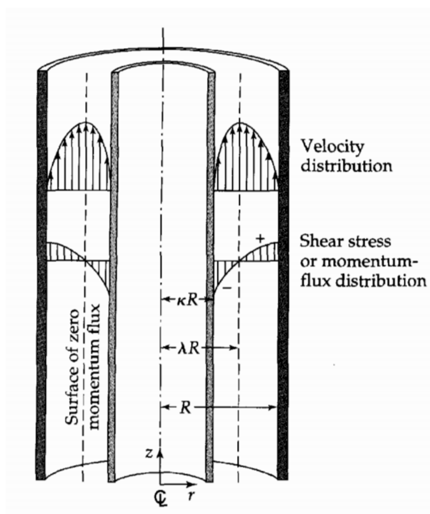
[3]The modified pressure, $\mathscr{P}$, is defined as $\mathscr{P} = P + \rho g h$ where h is the distance in the direction opposite of gravity. In this problem, $\mathscr{P} = P - \rho g z$ since height z is in the same direction as gravity

7. Integrating the above equations yields $v_z = -\left(\frac{\mathscr{P}_0 - \mathscr{P}_L}{4\mu L}\right) r^2 + C_2$, and $C_2 = \frac{(\mathscr{P}_0 - \mathscr{P}_L) R^2}{4\mu L}$ since the solid-liquid boundary condition states that $v_z = 0$ at $r = R$, so

$$v_z = \frac{(\mathscr{P}_0 - \mathscr{P}_L) R^2}{4\mu L}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

3.5 Flow Through an Annulus

Consider the following system where the fluid is moving upward in an annulus of height L (graphic from BSL)



1. This problem is best done using cylindrical coordinates. The fluid is moving in the z direction, so the momentum balance is in the z direction with a differential in the r direction. $v_z(r)$, so the only relevant stress is τ_{rz}. Also, $P(z)$, and the force of gravity is negative since it's in the opposite direction of fluid flow

2. The shell balance can be written as

$$2\pi r \Delta r P_0 - 2\pi r \Delta r P_L + 2\pi r \tau_{rz}\big|_r - 2\pi r L \tau_{rz}\big|_{r+\Delta r} - \rho 2\pi r L \Delta r g = 0$$

3. Dividing by $2\pi L \Delta r$ yields

$$\frac{-r\left(\tau_{rz}\big|_{r+\Delta r} - \tau_{rz}\big|_r\right)}{\Delta r} + \frac{r\left(P_0 - P_L\right)}{L} - \rho g r = 0$$

4. Letting the limit of Δr approach zero, and applying the definition of the derivative yields

$$\frac{d\left(r\tau_{rz}\right)}{dr} = \frac{r\left(P_0 - P_L - \rho g\right)}{L}$$

5. Substituting the modified pressure yields[4]

$$\frac{d\left(r\tau_{rz}\right)}{dr} = \left(\frac{\mathscr{P}_0 - \mathscr{P}_L}{L}\right) r$$

[4]Note here that $\mathscr{P} = P + \rho g z$ since the z height is in the opposite direction of gravity

6. Integrating the above equation yields

$$\tau_{rz} = \left(\frac{\mathscr{P}_0 - \mathscr{P}_L}{2L}\right) r + \frac{C_1}{r}$$

7. The constant of integration cannot be determined yet since we don't know the boundary momentum flux conditions. We know that the velocity only changes in the r direction, so there must be a maximum velocity at some arbitrary width $r = \lambda R$, and at this point there will be no stress. This is because τ_{ij} is a function of the rate of change of velocity, and since the derivative is zero at a maximum, the stress term will go to zero. Therefore,

$$0 = \left(\frac{\mathscr{P}_0 - \mathscr{P}_L}{2L}\right) \lambda R + \frac{C_1}{\lambda R}$$

8. Solving the above equation for C_1 and substituting it in yields

$$\tau_{rz} = \frac{(\mathscr{P}_0 - \mathscr{P}_L) R}{2L} \left[\left(\frac{r}{R}\right) - \lambda^2 \left(\frac{R}{r}\right)\right]$$

9. Using $\tau_{rz} = -\mu \frac{dv_z}{dr}$ and integrating $\frac{dv_z}{dr}$ yields

$$v_z = -\frac{(\mathscr{P}_0 - \mathscr{P}_L) R^2}{4\mu L} \left[\left(\frac{r}{R}\right)^2 - 2\lambda^2 \ln\left(\frac{r}{R}\right) + C_2\right]$$

10. The boundary conditions state that $v_z = 0$ at $r = \kappa R$ and $v_z = 0$ at $r = R$ (solid-liquid interfaces), which yields a system of equations that can be solved to yield $C_2 = -1$ and $\lambda^2 = \dfrac{1-\kappa^2}{2\ln\left(\frac{1}{\kappa}\right)}$

11. Substituting the results from above yields the general equations of

$$\tau_{rz} = \frac{(\mathscr{P}_0 - \mathscr{P}_L) R}{2L} \left[\left(\frac{r}{R}\right) - \frac{1-\kappa^2}{2\ln\left(\frac{1}{\kappa}\right)} \left(\frac{R}{r}\right)\right]$$

$$v_z = -\frac{(\mathscr{P}_0 - \mathscr{P}_L) R^2}{4\mu L} \left[1 - \left(\frac{r}{R}\right)^2 - \frac{1-\kappa^2}{\ln\left(\frac{1}{\kappa}\right)} \ln\left(\frac{r}{R}\right)\right]$$

4 Mass, Energy, and Momentum Balances

4.1 Mass and Energy Balance

- For mass balances,

$$\sum \dot{m}_{in} - \sum \dot{m}_{out} = \frac{d}{dt}(m)_{\text{system}} = \frac{d}{dt}(\rho V)_{\text{system}}$$

- For energy balances with heat (q) and work[5] (w),

$$q - w + \sum E_{in} - \sum E_{out} = \frac{d}{dt}(E_{\text{sys}})$$

- Therefore, a general equation can be written as the following where $\hat{e}$ is the internal energy per unit mass,

$$\sum m_{in}\left(\hat{e} + \frac{P}{\rho} + gz + \frac{1}{2}v^2\right)_{in} - \sum m_{out}\left(\hat{e} + \frac{P}{\rho} + gz + \frac{1}{2}v^2\right)_{out} + q - w = \frac{d}{dt}\left[m_{\text{sys}}\left(\hat{e} + gz + \frac{1}{2}v^2\right)\right]_{\text{sys}}$$

- Power can be written as

$$\text{Power} = \dot{m}\hat{w} = Fv = Q\Delta P$$

4.2 Bernoulli Equation

- At steady state conditions, the general equation for the mass-energy balance of an inlet-outlet system can be written as the following

$$\hat{e}_1 + \frac{v_1^2}{2} + gz_1 + \frac{P_1}{\rho_1} + q = \hat{e}_2 + \frac{v_2^2}{2} + gz_2 + \frac{P_2}{\rho_2} + \hat{w}$$

- The energy balance simplifies to the mechanical energy balance with constant g and at steady-state with $\hat{\mathcal{F}}$ representing frictional losses:

$$\Delta\left(\frac{v^2}{2}\right) + g\Delta z + \int_1^2 \frac{dP}{\rho} + \hat{w} + \hat{\mathcal{F}} = 0$$

  - $\hat{\mathcal{F}}$ is always positive, $\hat{w}$ is positive if the fluid performs work on the environment, and $\hat{w}$ is negative if the system has work done on it
  - Note that $\hat{\mathcal{F}}$ is a frictional energy per unit mass. Recall that energy is $\frac{\text{kg m}^2}{\text{s}^2}$. Also, $\hat{w}$ is energy
    * Therefore, if one solves for $\hat{\mathcal{F}}$, this value can be divided by g to get the "friction head," which has SI units of m

- If the fluid is incompressible, ρ is constant, so it becomes

$$\Delta\left(\frac{v^2}{2}\right) + g\Delta z + \frac{\Delta P}{\rho} + \hat{w} + \hat{\mathcal{F}} = 0$$

- For steady-state, no work, no frictional losses, constant g, and constant density (incompressible), the Bernoulli Equation is obtained:

$$\Delta\left(\frac{v^2}{2}\right) + g\Delta z + \frac{\Delta P}{\rho} = 0$$

  - For a problem that involves draining, if the cross sectional area of the tank is significantly larger than the siphon or draining hole, v_1 can be approximated as zero

[5] A positive work value indicates work done *by* the system while a negative work indicates work done *on* the system

4.3 Momentum Balance

Of course, the momentum balance was already seen in the shell momentum balance section, but for clarity it is included here as well (without assuming steady state conditions):

- Recall that momentum is:

$$\mathcal{M} \equiv mv$$

- Therefore,

$$\sum_{in} (\dot{m}v) - \sum_{out} (\dot{m}v) + \sum_{sys} F = \frac{d}{dt} (mv)_{sys} \equiv \frac{d\mathcal{M}_{sys}}{dt}$$

  – While it may seem impossible to add a force and momentum together, realize that is not what is being done. Instead, it's a mass flow rate times velocity, which happens to have the same units of force, so addition can be performed

- Some forces that are relevant to fluid dynamics are

$$\sum_{sys} F = \sum_{\text{gravity}} F + \sum_{\text{pressure}} F + \sum_{\text{visc.}} F + \sum_{\text{other}} F$$

5 Differential Equations of Fluid Mechanics[6]

5.1 Vectors and Operators

5.1.1 Dot Product

- If $\overrightarrow{u} = \langle u_1, u_2, u_3 \rangle$ and $\overrightarrow{v} = \langle v_1, v_2, v_3 \rangle$, then the dot product is

$$\overrightarrow{u} \cdot \overrightarrow{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

5.1.2 Cross Product

- The cross product is the following where θ is between 0 and π radians

$$\overrightarrow{u} \times \overrightarrow{v} = (u_2 v_3 - u_3 v_2)\, \hat{i} - (u_1 v_3 - u_3 v_1)\, \hat{j} + (u_1 v_2 - u_2 v_1)\, \hat{k}$$

5.1.3 Gradient

- The ∇ operator is defined as

$$\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$$

- The gradient of an arbitrary scalar function $f(x, y, z)$ is

$$grad(f) = \nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}$$

5.1.4 Divergence

- For an arbitrary 3-D vector $\overrightarrow{v}$, the divergence is

$$div\left(\overrightarrow{v}\right) = \nabla \cdot \overrightarrow{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

[6] See **Section 10** for equations written out for the various coordinate systems

5.1.5 Curl

- The curl of an arbitrary 3-D vector $\overrightarrow{v}$ is

$$curl(\overrightarrow{v}) = \nabla \times \overrightarrow{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{k}$$

5.1.6 Laplacian

- The Laplacian operator, ∇^2, acting on an arbitrary scalar $f(x, y, z)$ is

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

5.2 Solution of the Equations of Motion

- A differential mass balance known as the continuity equation can be set up as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \overrightarrow{v} = 0$$

- If ρ is constant, then $\frac{\partial \rho}{\partial t} = 0$, and the ρ values can be factored out of the above equation to make

$$\nabla \cdot \overrightarrow{v} = 0$$

- For constant ρ and μ, the Navier-Stokes Equation states that

$$\rho\left(\frac{\partial \overrightarrow{v}}{\partial t} + \overrightarrow{v} \cdot \nabla \overrightarrow{v}\right) = -\nabla P + \mu \nabla^2 \overrightarrow{v} + \rho \overrightarrow{g}$$

5.3 Procedure for Using Navier-Stokes Equation

1. Choose a coordinate system and find the direction of the flow
2. Use the continuity equation (making simplifications if ρ is constant) to find out more information
3. Use the Navier-Stokes equation in the direction of the fluid flow and eliminate terms that are zero based on the direction of the flow and the results of the continuity equation
   (a) Be careful to make sure that $\overrightarrow{g}$ is in the correct direction
4. Integrate the resulting equation and solve for the boundary conditions

5.4 Flow Through a Circular Tube Using Navier-Stokes

The problem in **Section 3.4** can be revisited, as the Navier-Stokes equation can be used on problems where the shell momentum balance method was originally used. For this problem, refer to the earlier diagram, and assume the pressure drops linearly with length. Also assume that ρ and μ are constant:

1. The best coordinate system to use is cylindrical, and the continuity equation can be written as

$$\nabla \cdot \overrightarrow{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{d\theta} + \frac{\partial v_z}{\partial z} = 0$$

2. The continuity equation turns into the following since velocity is only in the z direction:

$$\frac{\partial v_z}{\partial z} = 0$$

3. Now the Navier-Stokes equation can be written in the z direction as

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{d\theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right) + \rho g_z$$

4. The Navier-Stokes equation simplifies to the following since $v_z(r)$, $v_r = v_\theta = 0$, and the linear change in pressure means $\frac{\partial P}{\partial z} = \frac{P_L - P_0}{L}$:

$$0 = -\frac{P_L - P_0}{L} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \rho g_z$$

5. Rearranging the above equation yields

$$\frac{1}{\mu}\left[\frac{P_L - P_0}{L} - \rho g_z\right] = \frac{1}{r}\frac{d}{dr}\left[r\frac{\partial v_z}{\partial r}\right]$$

6. For simplicity's sake, let the left-hand side equal some arbitrary constant α to make integration easier

$$\alpha = \frac{1}{r}\frac{d}{dr}\left[r\frac{\partial v_z}{\partial r}\right]$$

7. Integrating the above equation yields

$$\frac{\alpha r^2}{2} = r\frac{\partial v_z}{\partial r} + C_1$$

8. Integrating the above equation again yields

$$\frac{\alpha r^2}{4} - C_1 \ln r + C_2 = v_z$$

9. At $r = 0$, the log term becomes unphysical, so $C_1 = 0$. The equation is now

$$\frac{\alpha r^2}{4} + C_2 = v_z$$

10. There is a solid-liquid boundary at $r = R$, and $v_z = 0$ here, so $C_2 = -\frac{\alpha R^2}{4}$. Therefore,

$$v_z = \frac{\alpha r^2}{4} - \frac{\alpha R^2}{4} = \alpha\left(r^2 - R^2\right) = \frac{1}{4\mu}\left[\frac{P_L - P_0}{L} - \rho g_z\right]\left(r^2 - R^2\right)$$

11. Substituting $\mathscr{P} = P - \rho g z$ into this problem yields the same result as in **Section 3.4**

5.5 Flow Through a Heat-Exchanger

Consider a heat-exchanger as shown below with an inner radius of R_1 and outer radius of R_2 with z pointing to the right. Fluid is moving to the right $(+z)$ in the outer tube, and the fluid is moving to the left in the inner tube (-z). Assume that the pressure gradient is linear and that $R_1 < r < R_2$:

1. Cylindrical coordinates are best to use here, and velocity is only in the z direction. The continuity equation is equivalent to the following, assuming constant ρ and μ

$$\nabla \cdot \overrightarrow{v} = 0 \rightarrow \frac{\partial v_z}{\partial z} = 0$$

2. Setting up the Navier-Stokes equation in the z direction simplifies to the following (recall that ρg isn't necessary since gravity isn't playing a role)

$$\frac{\partial P}{\partial z} = \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) \right]$$

3. Since the pressure gradient is linear,

$$\frac{P_L - P_0}{\mu L} r = \frac{d}{dr} \left(r \frac{dv_z}{dr} \right)$$

4. Integrating the above equation yields

$$\frac{P_L - P_0}{2\mu L} r^2 = r \frac{dv_z}{dr} + C_1$$

5. Integrating the above equation yields

$$v_z = \frac{P_L - P_0}{4\mu L} r^2 - C_1 \ln r + C_2$$

6. Letting the constant term in front of the r^2 become α for simplicity yields

$$v_z = \alpha r^2 - C_1 \ln r + C_2$$

7. At $r = R_1$, $v_z = 0$ since it's a solid-liquid boundary[7], so

$$C_2 = -\alpha R_1^2 + C_1 \ln R_1$$

8. At $r = R_2$, $v_z = 0$ since it's a solid-liquid boundary, so

$$C_1 = \frac{-\alpha \left(R_2^2 - R_1^2 \right)}{\ln \left(\frac{R_1}{R_2} \right)}$$

9. Substituting back into the equation for v_z yields

$$v_z = \alpha \left[r^2 + \frac{\left(R_2^2 - R_1^2 \right)}{\ln \left(\frac{R_1}{R_2} \right)} \ln r - R_1^2 - \frac{\ln R_1 \left(R_2^2 - R_1^2 \right)}{\ln \left(\frac{R_1}{R_2} \right)} \right]$$

10. Substituting back in for α and simplifying yields

$$v_z = \frac{P_L - P_0}{4\mu L} \left[\frac{\ln \left(\frac{r}{R_2} \right)}{\ln \left(\frac{R_2}{R_1} \right)} \left(R_2^2 - R_1^2 \right) + \left(R_2^2 - r^2 \right) \right]$$

[7]Note that $C_1 \neq 0$ because r is never actually zero. The outer tube cannot be smaller than the inner tube, so zero volume in the outer tube is actually $r = R_1$

6 Velocity Distributions with More Than One Variable

6.1 Time-Dependent Flow of Newtonian Fluids

6.1.1 Definitions

- To solve these problems, a few mathematical substitutions will be made and should be employed when relevant to make the final solution simpler
- The kinematic viscosity is
$$\nu = \frac{\mu}{\rho}$$
- For a wall-bounded flow, the dimensionless velocity is defined as the following where v_i is the local velocity and v_0 is the friction velocity at the wall
$$\phi = \frac{v_i}{v_0}$$

6.1.2 Flow near a Wall Suddenly Set in Motion

A semi-infinite body of liquid with constant density and viscosity is bounded below by a horizontal surface (the xz-plane). Initially, the fluid and the solid are at rest. Then at $t = 0$, the solid surface is set in motion in the $+x$ direction with velocity v_0. Find v_x, assuming there is no pressure gradient in the x direction and that the flow is laminar.

1. Cartesian coordinates should be used. Also, $v_y = v_z = 0$, and $v_x = v_x(y, t)$ since it will change with height and over time
2. Using the continuity equation yields $\frac{\partial v_x}{\partial x} = 0$ at constant ρ, which we already know because v_x is not a function of x
3. Using the Navier-Stokes equation in the x direction yields $\rho\left(\frac{\partial v_x}{\partial t}\right) = \mu\left(\frac{\partial^2 v_x}{\partial y^2}\right)$. This can be simplified to $\frac{\partial v_x}{\partial t} = \nu\frac{\partial^2 v_x}{\partial y^2}$
4. Boundary and initial conditions must be set up:
   (a) There is a solid-liquid boundary in the x direction, and the wall is specified in the problem stating as moving at v_0. Therefore, at $y = 0$ and $t > 0$, $v_x = v_0$
   (b) At infinitely high up, the velocity should equal zero, so at $y = \infty$, $v_x = 0$ for all $t > 0$
   (c) Finally, time is starting at $t = 0$, so $v_x = 0$ at $t \leq 0$ for all y
5. It is helpful to have the initial conditions cause solutions to be values of 1 or 0, so the dimensionless velocity of $\phi(y, t) = \frac{v_x}{v_0}$ will be introduced such that $\frac{\partial \phi}{\partial t} = \nu\frac{\partial^2 \phi}{\partial y^2}$
   (a) It is now possible to say $\phi(y, 0) = 0$, $\phi(0, t) = 1$, and $\phi(\infty, t) = 0$
6. Since ϕ is dimensionless, it must be related to $\frac{y}{\sqrt{\nu t}}$ since this (or multiplicative scale factors of it) is the only possible dimensionless group from the given variables. Therefore, $\phi = \phi(\eta)$ where $\eta = \frac{y}{\sqrt{4\nu t}}$. The $\sqrt{4}$ term is included in the denominator for mathematical simplicity later on but is not necessary
7. With this new dimensionless quantity, the equation in Step 5 can be broken down from a PDE to an ODE

(a) First, $\frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial \eta}\frac{\partial \eta}{\partial t}$. The value for $\frac{\partial \eta}{\partial t}$ can be found from taking the derivative with respect to t of η defined in Step 6. This yields $\frac{\partial \phi}{\partial t} = -\frac{d\phi}{d\eta}\frac{1}{2}\frac{\eta}{t}$

(b) Next, $\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial \eta}\frac{\partial \eta}{\partial y}$. The value for $\frac{\partial \eta}{\partial y}$ can be found from taking the derivative with respect to y of η defined in Step 6. This yields $\frac{\partial \phi}{\partial y} = \frac{d\phi}{d\eta}\frac{1}{\sqrt{4\nu t}}$

i. We want $\frac{\partial^2 \phi}{\partial y^2}$ though, so perform $\frac{\partial^2 \phi}{\partial y^2} = \frac{\partial \phi}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) = \left(\frac{d\phi}{d\eta}\cdot\frac{1}{\sqrt{4\nu t}}\right)\cdot\left(\frac{d\phi}{d\eta}\cdot\frac{1}{\sqrt{4\nu t}}\right) = \frac{d^2\phi}{d\eta^2}\frac{1}{4\nu t}$

(c) Therefore, $\frac{d^2\phi}{d\eta^2} + 2\eta\frac{d\phi}{d\eta} = 0$

8. New sets of boundary conditions are needed for η

(a) At $\eta = 0$, $\phi = 1$ since this is when $y = 0$, and it was stated earlier that $\phi(0, t) = 1$

(b) At $\eta = \infty$, $\phi = 0$ since this is when $y = \infty$, and it was stated earlier that $\phi(\infty, t) = 0$

9. To solve this differential equation, introduce $\psi = \frac{d\phi}{d\eta}$ to make the equation $\frac{d^2\phi}{d\eta^2} + 2\eta\psi = 0$, which will yield $\psi = c_1 \exp\left(-\eta^2\right)$

10. Integrating ψ yields $\phi = c_1 \int_0^\eta \exp\left(-\bar{\eta}^2\right) d\bar{\eta} + c_2$

(a) $\bar{\eta}$ is used here since it is a dummy variable of integration and should not be confused with the upper bound of η

11. The boundary conditions of $\phi = 0$ and $\phi = 1$ can be used here to find c_1 and c_2, which produces the equation $\phi(\eta) = 1 - \operatorname{erf}(\eta)$

(a) The error function is defined for an arbitrary z and ξ as $\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-\xi^2} d\xi$

6.2 The Potential Flow and Streamfunction

- The vorticity of a fluid is defined as

$$\vec{w} = \nabla \times \vec{v}$$

  - If the vorticity of a fluid is zero, then it is said to be irrotational

- The velocity potential, ϕ, can be defined as the following, which satisfies irrotationality[8]

$$\vec{v} = -\nabla\phi$$

- The stream function, ψ, can be defined as the following, which satisfies continuity[9]

$$\vec{v} = \hat{z} \times \nabla\psi$$

- The Laplace equation is

$$\nabla^2\phi = 0$$

  - To obtain the Laplace equation from ϕ, substitute it into the continuity equation
  - To obtain the Laplace equation from ψ, substitute it into the irrotationality condition

- A stagnation point is defined as the point where the velocity in all dimensions is zero

[8] A table of velocity potentials can be found in the **Appendix**.

[9] A table of stream functions can be found in the **Appendix**.

6.3 Solving Flow Problems Using Streamfunctions

6.3.1 Overview

- To solve two-dimensional flow problems, the streamfunction can be used, and the equations listed in the **Appendix** for the differential equations of ψ that are equivalent to the Naiver-Stokes equation should be used
- For steady, creeping flow, the only term not equal to zero is the term associated with the kinematic viscosity, ν
  - This is also true at Re $\ll 1$
- When dealing with spherical coordinates, r and θ will still be used, but z will be used in place of ϕ for the third dimension

6.3.2 Creeping Flow Around a Sphere[10]

Problem Statement: Obtain the velocity distributions when the fluid approaches a sphere in the positive z direction (if z is to the right). Assume that the sphere has radius R and Re $\ll 1$.

Solution:

First, realize that this is two-dimensional flow, so a stream function should be used in place of the Navier-Stokes equation for (relative) mathematical simplicity. The ψ equivalent for the Navier-Stokes equation in spherical coordinates can be found in the **Appendix**. Since Re $\ll 1$, the stream function differential equation becomes $0 = \nu E^4 \psi$, which is simplified to $0 = E^4\psi$. Substituting in $\left(E^2\right)^2$ into this equation yields

$$\left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2 \psi = 0 \tag{1}$$

The next step is to find the boundary conditions. Note that it is best to find all boundary conditions regardless of how many variables are actually needed. The first two boundary conditions are for the no-slip solid-liquid boundary condition where $r = R$. Here,

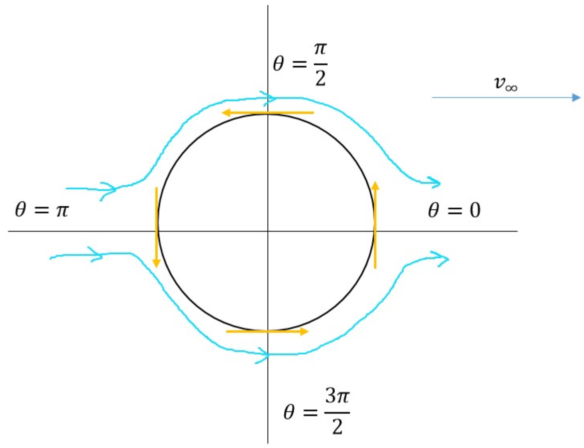
$$v_r = -\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} = 0 \qquad (r = R) \tag{2}$$

and

$$v_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} = 0 \qquad (r = R) \tag{3}$$

which are the definitions of the stream function in spherical coordinates. The velocity components at $r = R$ equal zero since the sphere itself is not moving. For the last boundary condition, analyze the system at a point far away from the sphere: $r \to \infty$. To do this, first note that v_θ is tangential to the streamlines around the sphere. Due to mathematical convention (quadrants are numbered counterclockwise), this is graphically shown as

[10] This problem has not been explained in full detail with regards to both concepts and the math in all the textbooks and websites I could find, so effort has gone into a thorough explanation here.



The diagram shown above indicates the following. The projection of the sphere (a circle) is drawn such that it is similar to a unit circle with the angles (θ) drawn in at the appropriate quadrants. The cyan lines represent the streamlines. v_∞ represents some velocity at a point infinitely far out to the right. The yellow vectors represent the direction of v_θ, which is always tangential to the sphere and goes counterclockwise. Note how the yellow v_θ vector goes in the same direction of v_∞ at $\theta = \frac{3\pi}{2}$ and against the direction of v_∞ at $\theta = \frac{\pi}{2}$. Also notice that v_θ is completely perpendicular to v_∞ at $\theta = 0, \pi$.

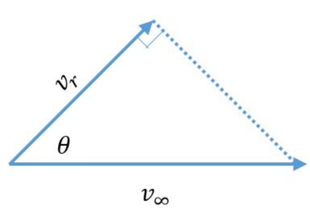
With this diagram available, one should find v_θ at each quadrant when $r \to \infty$. At $\theta = 0$, $v_\theta = 0$ since the flow of the fluid is perpendicular to v_θ and thus none of the fluid is actually flowing in the v_θ direction. At $\theta = \frac{\pi}{2}$, $v_\theta = -v_\infty$ since the flow is parallel (but in the opposite direction) to v_θ, assuming that v_∞ is the velocity of the fluid infinitely far to the right of the sphere. At $\theta = \pi$, $v_\theta = 0$ since the flow is perpendicular to v_θ. At $\theta = \frac{3\pi}{2}$, $v_\theta = v_\infty$ since the flow is parallel (and in the same direction) to v_θ.

The next step is to come up with a function for v_θ given the values we previously defined. The most reasonable function that has the values of v_θ shown earlier at each axis is

$$v_\theta = -v_\infty \sin\theta \tag{4}$$

This can be seen by analyzing the sine function. Sine goes from 0 to 1 to 0 to -1 in intervals of $\frac{\pi}{2}$ around the unit circle. Multiplying sine by a factor of $-v_\infty$ simply changes the magnitude of the function and ensures the correct value at each point.

As a quick aside, one might care to know what v_r is equal to at $r \to \infty$ since we have just found v_θ at $r \to \infty$ (Eq. 4). To do this, analyze the vector triangle shown below. Note that v_∞ is horizontal since it is defined as the velocity infinitely far out to the right of the sphere, and v_r is at some arbitrary angle θ since it represents a radial value, which can be oriented at infinitely many angles. The triangle is therefore



Using trigonometry, one can state the following (note the location of the right angle and thus the hypotenuse):

$$v_r = v_\infty \cos\theta$$

With (Eq. 4), we can equate v_θ to the corresponding definition of the stream function. Recall that $v_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$ in spherical coordinates. This can be rearranged to $d\psi = -v_\infty r \sin^2\theta dr$ by cross-multiplication and substitution of (Eq. 4) for v_θ. This can then be integrated to yield the third and final boundary condition of

$$\psi = -\frac{1}{2}v_\infty r^2 \sin^2\theta \qquad (r \to \infty) \tag{5}$$

A solution must now be postulated for ψ. To do this, look at (Eq. 5). It is clear that ψ has an angular component that is solely a function of $\sin^2\theta$. There is also a radial component, which can be assigned some arbitrary $f(r)$. Therefore, a postulated solution is of the form

$$\psi(r,\theta) = f(r)\sin^2\theta \tag{6}$$

Now, substitute (Eq. 6) into (Eq. 1) and solve. This is a mathematically tedious process, but the bulk of the work has been shown below. Recognize that it is probably easiest to find $E^2\psi$ and then do $E^2\left(E^2\psi\right)$ instead of trying to do $E^4\psi$ all in one shot. Therefore,

$$E^2\psi = \left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right] f(r)\sin^2\theta = 0$$

Multiply $f(r)\sin^2\theta$ through and be careful of terms that are and are not influenced by the partial derivatives. Also recall that differential operators are performed from right to left. This yields

$$E^2\psi = \sin^2\theta\frac{d^2 f(r)}{dr^2} + \frac{f(r)\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\cdot 2\cos\theta\sin\theta\right) = 0$$

After applying the right-most derivative,

$$E^2\psi = \sin^2\theta\frac{d^2 f(r)}{dr^2} - \frac{2f(r)\sin^2\theta}{r^2} = 0$$

Factoring $\sin^2\theta$ out of the equation yields

$$E^2\psi = \frac{d^2 f(r)}{dr^2} - \frac{2f(r)}{r^2} = 0 \tag{7}$$

This is close to the final solution of $E^4\psi = 0$, but recall that this was just $E^2\psi$ that was evaluated. We must now evaluate $E^2\left(E^2\psi\right)$, or, equivalently, E^2 of (Eq. 7). This is equivalent to saying

$$E^4\psi = \left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]\left[\frac{d^2 f(r)}{dr^2} - \frac{2f(r)}{r^2}\right] = 0$$

This simplifies to

$$E^4\psi = \left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right) f = 0 \tag{8}$$

The above differential equation is referred to as an equidimensional equation, and equidimensional equations should be tested with trial solutions of the form Cr^n. An equidimensional equation has the same units throughout. For instance, $\frac{d^2}{dr^2}$ has units of m^{-2} and so does $-\frac{2}{r^2}$. Therefore, since all terms have the same units, this is an equidimensional equation. By substituting Cr^n into (Eq. 8), n may have values of -1, 1, 2, and 4. This can be seen by the following:

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right) Cr^n = 0$$

Applying the right-hand parenthesis yields

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right) n\,(n-1)\,Cr^{n-2} - 2Cr^{n-2} = 0$$

This simplifies to

$$\left(\frac{d^2}{dr^2}-\frac{2}{r^2}\right)Cr^{n-2}\left[n\left(n-1\right)-2\right]=0$$

Applying the left-hand parentheses yields

$$\left(n-2\right)\left(n-3\right)Cr^{n-4}\left[n\left(n-1\right)-2\right]-2Cr^{n-4}\left[n\left(n-1\right)-2\right]$$

Simplifying this yields

$$\left[\left(n-2\right)\left(n-3\right)-2\right]\left[n\left(n-1\right)-2\right]Cr^{n-4}=0$$

Finally, one more algebraic simplification yields

$$\left(n-4\right)\left(n-2\right)\left(n-1\right)\left(n+1\right)Cr^{n-4}=0$$

The solutions to the above equation are $n=-1,1,2,4$. This makes $f(r)$ the following general equation with four terms:

$$f(r)=C_1r^{-1}+C_2r+C_3r^2+C_4r^4 \tag{9}$$

The constant C_4 must equal zero. This is because C_4 is multiplied by r^4, and ψ (Eq. 5/6) does not have an r^4 term in it. C_3 must equal $-\frac{1}{2}v_\infty$ because C_3 is multiplied by an r^2 term, and the equation for ψ (Eq. 5/6) has a $-\frac{1}{2}v_\infty r^2$ in it. The reason that $C_4=0$ and C_1 and C_2 do not is that we know that the highest term in the equation for ψ is an r^2 term. Any term that converges faster than the maximum order (r^2) must be zero. Therefore, the C_1 term and C_2 term do not have to go to zero, but the C_4 term does. This makes the equation for ψ the following by plugging (Eq. 9) into (Eq. 6):

$$\psi(r,\theta)=\left(C_1r^{-1}+C_2r-\frac{1}{2}v_\infty r^2\right)\sin^2\theta \tag{10}$$

Using the definition of v_r in relation to the stream function, one can rewrite this equation as the following by plugging the equation for ψ (Eq. 10) into $\frac{\partial\psi}{\partial\theta}$ in the equation for v_r:

$$v_r=-\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}=\left(v_\infty-2\frac{C_2}{r}-2\frac{C_1}{r^3}\right)\cos\theta \tag{11}$$

Similarly for v_θ,

$$v_\theta=\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}=\left(-v_\infty+\frac{C_2}{r}-\frac{C_1}{r^3}\right)\sin\theta \tag{12}$$

By setting $v_r=0$ and $v_\theta=0$ at the $r=R$ no-slip boundary conditions and solving for C_1 and C_2, we get a system of 2 linear equations with $C_1=-\frac{1}{4}v_\infty R^3$ and $C_2=\frac{3}{4}v_\infty R$. One somewhat easy way to do this is to add (Eq. 11) and (Eq. 12), which cancels a lot of terms in the system of equations. However, any method to solve the system is sufficient. With these constants, one can rewrite the equation for ψ (Eq. 10) as

$$\psi\left(r,\theta\right)=\left(-\frac{1}{4}v_\infty\frac{R^3}{r}+\frac{3}{2}v_\infty Rr-\frac{1}{2}v_\infty r^2\right)\sin^2\theta \tag{13}$$

Therefore, rewriting (Eq. 11) and (Eq. 12) with the values for the constants substituted in yields the following

$$v_r=v_\infty\left(1-\frac{3}{2}\left(\frac{R}{r}\right)+\frac{1}{2}\left(\frac{R}{r}\right)^3\right)\cos\theta$$

$$v_\theta=-v_\infty\left(1-\frac{3}{4}\left(\frac{R}{r}\right)-\frac{1}{4}\left(\frac{R}{r}\right)^3\right)\sin\theta$$

6.4 Solving Flow Problems with Potential Flow

6.4.1 Overview

- To solve two-dimensional problems with fluids that have very low viscosities, are irrotational, are incompressible, and are at steady-state, the potential flow method can be used

6.4.2 Steady Potential Flow Around a Stationary Sphere

Problem Statement: Consider the flow of an incompressible, inviscid fluid in irrotational flow around a sphere. Solve for the velocity components.

Solution:

This problem is best done using spherical coordinates. The boundary conditions should then be found. There is a no-slip boundary condition at $r = R$. Here,

$$v_r = -\frac{\partial \phi}{\partial r} = 0 \qquad (r = R) \tag{14}$$

and

$$v_\theta = -\frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0 \qquad (r = R) \tag{15}$$

Now, infinitely far away should be analyzed. At this point,

$$v_r = v_\infty \cos\theta \qquad (r \to \infty) \tag{16}$$

and

$$v_\theta = -v_\infty \sin\theta \tag{17}$$

For explanation of how to obtain (Eq. 16/17), see the previous example problem with the stream function.

Now, analyze ϕ in order to come up with a trial expression. This is done by substituting (Eq. 16) or (Eq. 17) into (Eq. 14) or (Eq. 15) and solving for ϕ via integration. Whether you choose to substitute v_r or v_θ, the reasonable trial solution is of the form

$$\phi(r, \theta) = f(r)\cos\theta \tag{18}$$

Next, substitute (Eq. 18) into the Laplace equation such that

$$\nabla^2 f(r)\cos\theta = 0$$

This expression becomes the following in spherical coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \phi}{\partial \theta}\right) = 0$$

Doing some of the derivatives yields

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df(r)}{dr}\cos\theta\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(-f(r)\sin^2\theta\right) = 0$$

Performing the right-most derivative and factoring out the cosine yields the following. Note that the left-most derivative is retained since doing a product rule is just extra work.

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df(r)}{dr}\right) - \frac{2f(r)}{r^2} = 0 \tag{19}$$

Note that (Eq. 19) is an equidimensional equation (see previous example problem with the stream function for more detail). Therefore, a test solution of Cr^n should be inputted for $f(r)$ in (Eq. 19). Doing this yields

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\left(Cr^n\right)}{dr}\right) - \frac{2\left(Cr^n\right)}{r^2} = 0$$

This should be some fairly simple calculus, which will result in the algebraic expression

$$(n+2)(n-1)=0$$

This has roots of $n=-2$ and $n=1$, so the trial expression $f(r)$ can be written as

$$f(r)=C_1 r+C_2 r^{-2}$$

Substituting $f(r)$ into (Eq. 18) yields

$$\phi(r,\theta)=\left(C_1 r+C_2 r^{-2}\right)\cos\theta \tag{20}$$

Apply the boundary conditions to solve for the constants. For instance, at $r=R$, we know that $v_r=-\dfrac{\partial\phi}{\partial r}=0$, so substituting (Eq. 20) into this expression yields

$$C_1-\frac{2C_2}{r^3}=0$$

Evaluating this at $r=R$ and rearranging yields

$$C_2=\frac{1}{2}C_1 R^3 \tag{21}$$

At $r\to\infty$, we know that $v_r=v_\infty\cos\theta=-\dfrac{\partial\phi}{\partial r}$, so substituting (Eq. 20) into this expression yields

$$-\left(C_1-2C_2 r^{-3}\right)\cos\theta=v_\infty\cos\theta$$

Evaluating this at $r\to\infty$ and rearranging yields

$$C_1=-v_\infty$$

Plugging the above expression into (Eq. 21) will yield

$$C_2=-\frac{1}{2}v_\infty R^2$$

Plugging the newly found expressions for C_1 and C_2 into $f(r)$ yields

$$f(r)=-v_\infty\left(1+\frac{R^2}{2r^2}\right)$$

and substituting this $f(r)$ into (Eq. 20) yields

$$\phi(r,\theta)=-v_\infty\left(1+\frac{R^2}{2r^2}\right)\cos\theta \tag{22}$$

Now that an expression for ϕ is found, we can find the velocity components using $\overrightarrow{v}=-\nabla\phi$ (or the equivalent definitions in the **Appendix**). For instance, we know that

$$v_r=-\frac{\partial\phi}{\partial r},\qquad v_\theta=-\frac{1}{r}\frac{\partial\phi}{\partial\theta}$$

Therefore, plugging (Eq. 22) into the above expressions will yield v_r and v_θ after simplification. They will come to

$$v_r=v_\infty\left(1-\left(\frac{R}{r}\right)^3\right)\cos\theta$$

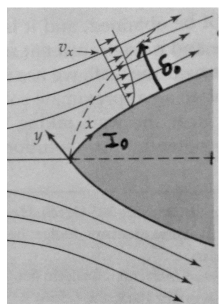
and

$$v_\theta=-v_\infty\left(1+\frac{1}{2}\left(\frac{R}{r}\right)^3\right)\sin\theta$$

6.5 Boundary Layer Theory

6.5.1 Derivation of Prandtl Boundary Layer Conditions

- Boundary layer theory involves the analysis of a very thin region (the boundary layer), which, due to its thinness, can be modeled in Cartesian coordinates despite any apparent curvature. For consistency, x will indicate downstream and y will indicate a direction perpendicular to a solid surface
- To be clear, a boundary layer is the following. Consider a solid object in a fluid. The area we are investigating is near this solid - not on the edge of the solid yet not very far away from the solid. Therefore, we will consider a hypothetical thin boundary layer that surrounds the solid
- Let v_∞ be the approach velocity on the surface (arbitrary dimension), I_0 be the length of the object, and δ_0 be the thickness of the thin boundary layer
  - Since we define this as a very thin boundary layer, we know that $\delta_0 \ll I_0$
- The scenario described above is depicted below for reference. The dashed line is the boundary layer (graphic from BSL):



- The continuity equation for this system is $\dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} = 0$, and the N-S equation in the x and y directions can be written as

$$\left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y}\right) = -\frac{1}{\rho}\frac{\partial \mathscr{P}}{\partial x} + \nu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right) \tag{23}$$

  and

$$\left(v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y}\right) = -\frac{1}{\rho}\frac{\partial \mathscr{P}}{\partial y} + \nu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2}\right) \tag{24}$$

- To solve a boundary layer problem, approximations need to be made that will simplify the N-S equations
- We know that $v_x = 0$ at the solid-liquid boundary (no-slip condition). Therefore, v_x varies from 0 to v_∞ from the solid's edge to the edge of the hypothetical boundary layer. Also, δ_0 is the thickness, which is in the y direction. As such[11],

$$\frac{\partial v_x}{\partial y} = \mathrm{O}\left(\frac{v_\infty}{\delta_0}\right) \tag{25}$$

- In the length direction, I_0, the fluid can only slow down once it hits the solid. Therefore, v_x has a maximum of v_∞ such that

$$\frac{\partial v_x}{\partial x} = \mathrm{O}\left(\frac{v_\infty}{I_0}\right) \tag{26}$$

[11] The O operator indicates "order of magnitude of."

- Integrating (Eq. 26) yields

$$v_x = \mathrm{O}\left(\frac{v_\infty}{I_0}\int_0^{I_0} dx\right) = \mathrm{O}\left(v_\infty\right)$$

- The continuity equation can be performed to find out that

$$\frac{\partial v_y}{\partial y} = \mathrm{O}\left(\frac{v_\infty}{I_0}\right) \tag{27}$$

  - Integrating (Eq. 27) yields

$$v_y = \mathrm{O}\left(\frac{v_\infty}{I_0}\int_0^{\delta_0} dy\right) = \mathrm{O}\left(\frac{v_\infty \delta_0}{I_0}\right)$$

    * This means that $v_y \ll v_x$ since we stated earlier that δ_0 is a very small quantity

- Looking at the N-S equations listed earlier, the terms can be replaced with their order of magnitude equivalents. First, the x direction N-S equation will be analyzed (Eq. 23):

  - First, the following relationship holds due to (Eq. 26) and the fact that $v_x = \mathrm{O}\left(v_\infty\right)$

$$v_x \frac{\partial v_x}{\partial x} = \mathrm{O}\left(\frac{v_\infty^2}{I_0}\right) \tag{28}$$

  - Second, the following relationship holds due to (Eq. 25) and the fact that $v_y = \mathrm{O}\left(\frac{v_\infty \delta_0}{I_0}\right)$ found earlier

$$v_y \frac{\partial v_x}{\partial y} = \mathrm{O}\left(\frac{v_\infty^2}{I_0}\right) \tag{29}$$

  - Next, the following relationship holds due to (Eq. 26):

$$\frac{\partial^2 v_x}{\partial x^2} = \mathrm{O}\left(\frac{v_\infty}{I_0^2}\right) \tag{30}$$

    * Note that the it is not v_∞^2 in the numerator because a second-derivative indicates two instances of dx and not two instances of v_x

  - This also means that the following relationship holds due to (Eq. 25):

$$\frac{\partial^2 v_x}{\partial y^2} = \mathrm{O}\left(\frac{v_\infty}{\delta_0^2}\right) \tag{31}$$

  - Additionally,

$$\frac{\partial^2 v_x}{\partial x^2} \ll \frac{\partial^2 v_x}{\partial y^2} \tag{32}$$

    * This is because we already stated $I_0 \gg \delta_0$, and δ_0 is in the denominator of (Eq. 31) while I_0 is in the denominator of (Eq. 30)

- Now all portions of the x direction N-S equation have been replaced piece-by-piece

- According to boundary layer theory, the left velocity components of N-S equations should have the same order of magnitude as the velocity components on the right side of the N-S equations at the boundary layer

- Therefore, rewriting (Eq. 23) with the previously defined order of magnitude analogues and the fact that $v_x \gg v_y$ as well as the relationship in (Eq. 32) yields

$$\frac{v_\infty^2}{l_0} = \mathrm{O}\left(\nu \frac{v_\infty}{\delta_0^2}\right)$$

- Rearranging the above equation and substituting in the Reynolds number yields the more frequently used relationship of

$$\frac{\delta_0}{l_0} = \mathrm{O}\left(\frac{1}{\sqrt{\mathrm{Re}}}\right)$$

- All information from the x direction N-S equation has now been extracted. To be completely mathematically rigorous, the y direction N-S equation should be analyzed, but this will simply yield that the N-S equation in the y direction is much less significant than the x direction N-S equation (see Eq. 4.4-9 in Bird for a mathematical justification). This does not really come as a surprise since we stated $v_y \ll v_x$ earlier. This also means that $\frac{\partial \mathscr{P}}{\partial y} \ll \frac{\partial \mathscr{P}}{\partial x}$ such that the modified pressure is only a function of x.

- Collectively, this information leads us to the Prandtl boundary layer equations, which are

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{33}$$

and

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{d\mathscr{P}}{dx} + \nu \frac{\partial^2 v_x}{\partial y^2} \tag{34}$$

- The first Prandtl boundary layer equation is simply the continuity equation stated at the beginning of this section, and the second equation is the simplified N-S equation in the x direction taking into account (Eq. 32)

6.5.2 Derivation of the von Karman Momentum Balance[12]

- With these assumptions in place, the Prandtl boundary layer equations can be solved for and will yield the Karman momentum balance

- Taking the derivative with respect to x of the Bernoulli Equation at the edge of the boundary will yield a useful relationship. Note that $v_x(x, y) \to v_e(x)$ at the outer edge of the boundary layer

$$\frac{d}{dx}\left[\frac{\mathscr{P}(x)}{\rho} + \frac{v_e(x)^2}{2}\right] = \frac{d}{dx}\,[\text{constant}]$$

- This simplifies to the following (note the use of product rule)

$$\frac{1}{\rho}\frac{d\mathscr{P}(x)}{dx} + \frac{1}{2}\left[v_e(x)\frac{dv_e(x)}{dx} + \frac{dv_e(x)}{dx}v_e(x)\right] = 0$$

- Further simplification yields

$$v_e(x)\frac{dv_e(x)}{dx} = -\frac{1}{\rho}\frac{d\mathscr{P}(x)}{dx} \tag{35}$$

- Now, (Eq. 33) will be analyzed. It can integrated and rearranged to yield[13]

$$v_y = -\int_0^y \frac{\partial v_x}{\partial x} d\bar{y} \tag{36}$$

[12] Not required for ChBE 21 at Tufts University.
[13] The "function of x" (e.g. $v_e(x)$) will be dropped for simplicity.

- Plugging (Eq. 35) and (Eq. 36) into (Eq. 34) yields

$$v_x \frac{\partial v_x}{\partial x} - \left(\int_0^y \frac{\partial v_x}{\partial x} d\bar{y} \right) \frac{\partial v_x}{\partial y} = v_e \frac{dv_e}{dx} + \nu \frac{\partial^2 v_x}{\partial y^2} \tag{37}$$

- Integrating (E. 37) from $y = 0$ to $y = \infty$ yields

$$\int_0^\infty \left[v_x \frac{\partial v_x}{\partial x} - \frac{\partial v_x}{\partial y} \left(\int_0^y \frac{\partial v_x}{\partial x} d\bar{y} \right) - v_e \frac{dv_e}{dx} \right] dy = \int_0^\infty \left[\nu \frac{\partial^2 v_x}{\partial y^2} \right] dy \tag{38}$$

  - The RHS of the above equation is

$$\int_0^\infty \nu \frac{\partial^2 v_x}{\partial y^2} dy = \nu \left. \frac{\partial v_x}{\partial y} \right|_0^\infty$$

    * Evaluating at $y = \infty$ yields zero since the top of the boundary layer should have inviscid outer flow (viscosity only matters in a region close to the boundary layer). This yields

$$\text{RHS} = -\frac{\mu}{\rho} \left. \frac{\partial v_x}{\partial y} \right|_{y=0} \tag{39}$$

  - The second term on the LHS can now be evaluated. It can be rewritten as

$$\text{Second Term on LHS} = -\int_0^\infty \frac{\partial v_x}{\partial y} \left[\int_0^y \frac{\partial v_x}{\partial x} d\bar{y} \right] dy$$

    * Integration by parts yields

$$\text{Second Term on LHS} = -\left[v_x|_0^\infty \int_0^\infty \frac{\partial v_x}{\partial x} dy - \int_0^\infty v_x \frac{\partial v_x}{\partial x} dy \right]$$

    * Recall that $v_x(x, y)$ and $v_x(x, \infty) \to v_e(x)$ such that the above equation can be rewritten as

$$\text{Second Term on LHS} = -\int_0^\infty v_e \frac{\partial v_x}{\partial x} dy + \int_0^\infty v_x \frac{\partial v_x}{\partial x} dy \tag{40}$$

  - Substituting (Eq. 39) and (Eq. 40) into (Eq. 38) yields

$$2 \int_0^\infty v_x \frac{\partial v_x}{\partial x} dy - \int_0^\infty v_e \frac{\partial v_x}{\partial x} dy - \int_0^\infty v_e \frac{dv_e}{dx} dy = -\frac{\mu}{\rho} \left. \frac{\partial v_x}{\partial y} \right|_{y=0} \tag{41}$$

  - After some manipulation and multiplication by ρ, the above equation becomes the von Karman momentum balance of

$$\mu \left. \frac{\partial v_x}{\partial y} \right|_{y=0} = \frac{d}{dx} \int_0^\infty \rho v_x (v_e - v_x) dy + \frac{dv_e}{dx} \int_0^\infty \rho (v_e - v_x) dy$$

7 Flow in Chemical Engineering Equipment

7.1 Laminar Flow

- For a horizontal pipe of length L and radius R, the velocity profile can be written as the following at any radial point r

$$v = \frac{-\Delta P}{4\mu L} \left(r^2 - R^2 \right)$$

- The total flow rate can be written as

$$Q = \int_0^{2\pi} \int_0^R v \, r dr d\theta = \frac{-\Delta P \pi R^4}{8\mu L}$$

- The mean velocity is given as

$$\langle v \rangle = \frac{Q}{\pi R^2} = \frac{-R^2 \Delta P}{8\mu L}$$

  and the maximum velocity is simply twice the mean velocity

7.2 Friction Factors for Flow in Tubes

- The Fanning friction factor is defined as the following with the usual definitions and τ_w as the wall shear stress

$$f_F = \frac{2\tau_w}{\rho\langle v\rangle^2} = \frac{-D\Delta\mathscr{P}}{2L\rho\langle v\rangle^2}$$

- For Laminar flow,

$$f_F = \frac{16}{\text{Re}}$$

7.3 Friction Factors for Flow Around Submerged Objects

- The coefficient of drag is given as

$$C_D = \frac{F_D/A_P}{\rho v_\infty^2}$$

  - A_P is the project area and is $\frac{\pi D^2}{4}$ for flow around a submerged sphere and F_D is the drag force and is given as $F_D = \frac{P}{v}$

- The settling of a spherical particle under the influence of gravity is given as

$$\frac{\pi D^3}{6}\left(\rho_s - \rho_f\right) g - F_D = \frac{\pi D^3 \rho_s}{6}\frac{dv}{dt}$$

- At terminal velocity or any other steady-state condition, $\frac{dv}{dt} = 0$. Therefore, solving for F_D and plugging into the definition of C_D yields

$$C_D = \frac{4gD\left(\rho_s - \rho_f\right)}{3v_\infty^2 \rho_f}$$

  - All subscripts of s indicate the spherical object and f for the fluid. For terminal velocity, $v_\infty = v_t$

- The above equation for C_D can be rewritten as

$$C_D\text{Re}^2 = \frac{4g\rho_f D^3}{3\mu^2}\left(\rho_s - \rho_f\right)$$

  or as

$$\frac{C_D}{\text{Re}} = \frac{4g\mu\left(\rho_s - \rho_f\right)}{3\rho_f^2 v_\infty^3}$$

8 Thermal Conductivity and the Mechanisms of Energy Transport

8.1 Fourier's Law of Heat Conduction

- The heat flux ($\overrightarrow{q}$) is defined as (Fourier's Law of Cooling)

$$\overrightarrow{q} = -k\nabla T$$

- The heat flux is related to the heat flow (Q) by

$$Q = \overrightarrow{q} A = -Ak\nabla T$$

- The thermal diffusivity is defined as the following where $\hat{C}_p$ is heat capacity at constant pressure with per-mass units,

$$\alpha = \frac{k}{\rho \hat{C}_p}$$

  - The units of α are length-squared per unit time

- The Prandtl number (unitless) is defined as

$$\text{Pr} = \frac{\nu}{\alpha} = \frac{\hat{C}_p \mu}{k}$$

8.2 The Microscopic Energy Balance

8.2.1 Equation

- The microscopic energy balance states that

$$\rho \hat{C}_p \left[\frac{\partial T}{\partial t} + \overrightarrow{v} \cdot \nabla T \right] = -\nabla \cdot \overrightarrow{q} + S$$

- Using Fourier's Law of cooling yields the equivalent[14]

$$\rho \hat{C}_p \left[\frac{\partial T}{\partial t} + \overrightarrow{v} \cdot \nabla T \right] = \nabla \left(k \nabla T \right) + S$$

  - Note that $\nabla \left(k \nabla T \right) = k \nabla^2 T$ when k is constant

- The $\frac{\partial T}{\partial t}$ is the accumulation term, the $\left(\overrightarrow{v} \cdot \nabla \right) T$ term is convective, the $\nabla \left(k \nabla T \right)$ is the conductive term, and S is the source term

- Dividing by $\rho \hat{C}_P$ and assuming k is constant yields

$$\left(\frac{\partial T}{\partial t} + \left(\overrightarrow{v} \cdot \nabla \right) T \right) = \alpha \nabla^2 T + \frac{S}{\rho \hat{C}_p}$$

  where

$$\alpha \equiv \frac{k}{\rho \hat{C}_P}$$

- Recall the following thermodynamic definition as well:

$$\hat{C}_p = \left(\frac{\partial H}{\partial T} \right)_P$$

8.2.2 Boundary Conditions

- The temperature may be specified at a surface
- The heat flux normal to a surface may be given (this is the same as saying the normal component of the temperature gradient)
- There must be continuity of temperature and heat flux normal to the surface at the interfaces
- At a solid-liquid interface, $q = h \left(T_0 - T_b \right)$ where T_b is the bulk temperature and T_0 is the solid surface temperature

[14] Since boundary conditions are more frequently based on temperature quantities, this form of the equation is typically more useful.

8.3 Conduction through a Block

Prompt: There is a rectangular prism with heat flow solely in the $+x$ direction. The left face of the object is at a temperature T_1, and the right face of the object is at a temperature T_2. The length of the box is $x = B$. Assume that $k = a + bT$, where a and b are constants. Solve for the heat flux through the object.

Solution:

1. The Cartesian coordinate system should be used. Also, there is only conduction, and it is in the x direction such that $T(x)$

2. Using the microscopic energy balance, realizing it is steady state so $\frac{\partial T}{\partial t} = 0$, that $\overrightarrow{v} = 0$, and that $S = 0$ yields

$$0 = \nabla \left(k \nabla T\right)$$

3. Substituting in for k and rewriting the equivalents for ∇ yields

$$0 = \frac{d}{dx}\left((a + bT)\frac{dT}{dx}\right)$$

4. Integrating once yields

$$C_1 = (a + bT)\frac{dT}{dx}$$

5. Integrating again yields

$$C_1 x + C_2 = aT + \frac{bT^2}{2}$$

6. At the interfaces, there must be continuity of temperature, so at $x = 0$, $T = T_1$, and at $x = B$, $T = T_2$

$$C_2 = aT_1 + \frac{bT_1^2}{2}$$

$$C_1 = \frac{a\left(T_2 - T_1\right)}{B} + \frac{b\left(T_2^2 - T_1^2\right)}{2B}$$

7. From this, the temperature profile can be fully described. However, since the heat flux is desired, Fourier's Law will be used. Note that step 4 indicated that $C_1 = k\frac{dT}{dx}$. Therefore,

$$q_x = -k\frac{dT}{dx} = -C_1 = -\left[\frac{a\left(T_2 - T_1\right)}{B} + \frac{b\left(T_2^2 - T_1^2\right)}{2B}\right]$$

8.4 Shell Energy Balance

- As with fluids, the shell balance can be used in place of the microscopic balance for heat flow. The equation is:

Convection In - Convection Out + Conduction In - Conduction Out + Work On System - Work By System + Rate of Energy Production = 0

- Conduction is given as Aq where A is the projected area (analogous to the stress term in the shell momentum balance)

- The rate of energy production is given as SV where S is the rate of heat production per unit volume and V is volume

8.5 Heat Conduction with an Electrical Heat Source

8.5.1 Shell Energy Balance

Problem: Find the temperature profile of a cylindrical wire with radius R, length L, an outside temperature of T_0, and a constant rate of heat production per unit volume of S_e.

Solution:

1. This problem is best done with cylindrical coordinates. Temperature is only a function of r, and the shell will become infinitesimally small in the radial direction

2. Setting up the shell energy balance with only conduction and a source yields

$$Aq\big|_{in} - Aq\big|_{out} + VS_e = 0$$

3. The conduction areas are the projection, which is the circumference times the length. The volume is simply the volume of the cylindrical shell

$$(2\pi rLq_r)\big|_r - (2\pi rLq_r)\big|_{r+\Delta r} + 2\pi rL\Delta rS_e = 0$$

4. Factoring out constants yields

$$2\pi L\left[(rq_r)\big|_r - (rq_r)\big|_{r+\Delta r}\right] + 2\pi rL\Delta rS_e = 0$$

5. Rearranging terms to make use of the definition of the derivative yields

$$2\pi L\left[(rq_r)\big|_{r+\Delta r} - (rq_r)\big|_r\right] - 2\pi rL\Delta rS_e = 0$$

6. Dividing by $2\pi L\Delta r$ and using the definition of the derivative yields

$$\frac{d\,(rq_r)}{dr} = rS_e$$

7. Integrating once yields

$$q_r = \frac{rS_e}{2} + \frac{C_1}{r}$$

8. There must be a value at $r = 0$, but at $r = 0$, the C_1 term blows up to infinity. Since this is non-physical, $C_1 = 0$ and the expression becomes

$$q_r = \frac{rS_e}{2}$$

9. Fourier's Law can now be used to introduce temperature such that

$$-k\frac{dT}{dr} = \frac{rS_e}{2}$$

10. Integrating once yields

$$T = -\frac{r^2S_e}{4k} + C_2$$

11. Since temperature must be continuous at the interface, at $r = R$, $T = T_0$. Therefore,

$$C_2 = T_0 + \frac{R^2S_e}{4k}$$

12. Rewriting the expression for temperature and simplifying yields

$$T - T_0 = \frac{S_eR^2}{4k}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

8.5.2 Microscopic Energy Balance

Problem: Repeat the previous example using the microscopic energy balance.
Solution:

1. Cylindrical coordinates are best used for this problem. Temperature is only a function of r, the system is at steady-state, the system is not moving, and k is assumed constant such that the microscopic energy balance becomes
$$0 = k\nabla^2 T + S_e$$

2. Substituting in for ∇^2 in cylindrical coordinates yields
$$k\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -S_e$$

3. Integrating once yields
$$r\frac{dT}{dr} = -\frac{S_e r^2}{2k} + C_1$$

4. Integrating a second time yields
$$T = -\frac{S_e r^2}{4k} + C_1 \ln|r| + C_2$$

5. Since the C_1 term becomes unphysical at $r = 0$, it must be true that $C_1 = 0$ such that
$$T = -\frac{S_e r^2}{4k} + C_2$$

6. The temperature must be continuous at the interface, so at $r = R$, $T = T_0$ such that
$$C_2 = T_0 + \frac{S_e R^2}{4k}$$

7. This makes the final expression for the temperature
$$T - T_0 = \frac{SR^2}{4k}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

8. To get the heat flux, use Fourier's Law at Step 3 such that
$$-\frac{q_r r}{k} = -\frac{S_e r^2}{2k} + C_1$$
   (a) This is because $q_r = -k\frac{dT}{dr}$

9. Since C_1 was found to be zero in Step 5,
$$-\frac{q_r r}{k} = -\frac{S_e r^2}{2k}$$

10. Simplifying yields
$$q_r = \frac{S_e r}{2}$$

8.6 Heat Conduction with a Nuclear Heat Source

Problem: Consider a double-layer spherical nuclear fuel element. There is a core spherical nuclear material surrounded by a spherical coating. The radius of the nuclear material is given as R_f, and the radius from the center to the outer edge of the coating is given as R_c. Find the heat flux and temperature profile, assuming that the source energy is not constant and is given as

$$S_n = S_{n0}\left[1 + b\left(\frac{r}{R_f}\right)^2\right]$$

where S_{n0} and b are constants. The outer temperature of the system is also at T_0.

Solution:

1. Spherical coordinates are best used here. The temperature and heat flux are only functions of r, and the spherical shell will have a thickness that approaches zero in the radial direction. The only terms are conduction and source for the heat flux from the nuclear fission. However, there is also another expression for the heat flux through the coating part. The two expressions will be identical, but the heat flux through the coating will not have a source.

2. Writing the shell energy balance for the fission heat flux yields with $A = 4\pi r^2$ (the surface area of a sphere) and $V = 4\pi r^2 \Delta r$,

$$4\pi\left[\left(r^2 q_r^f\right)\big|_r - \left(r^2 q_r^f\right)\big|_{r+\Delta r}\right] + 4\pi r^2 \Delta r S_n = 0$$

3. Dividing by $4\pi\Delta r$ and rearranging the terms to make use of the definition of the derivative yields

$$\frac{d\left(r^2 q_r^f\right)}{dr} = r^2 S_n$$

   (a) Since S_n is a function of r, the expression will need to be substituted in for proper integration. This yields

$$\frac{d\left(r^2 q_r^f\right)}{dr} = r^2 S_{n0}\left[1 + b\left(\frac{r}{R_f}\right)^2\right]$$

4. As stated before, the heat flux through the coating will be identical but without a source, so it is

$$\frac{d\left(r^2 q_r^c\right)}{dr} = 0$$

5. Integrating the equations in Step 3(a) and Step 4 independently yields

$$q_r^f = S_{n0}\left(\frac{r}{3} + \frac{br^3}{5R_f^2}\right) + \frac{C_1^f}{r^2}$$

$$q_r^c = \frac{C_1^c}{r^2}$$

6. The value of $r = 0$ is only applicable for the inner fissionable material (since the radial values of the coating are $r = R_f$ to $r = R_c$ instead of $r = 0$ to $r = R_f$). It is clear that $r = 0$, the C_1^f terms becomes unphysical, so $C_1^f = 0$ and thus

$$q_r^f = S_{n0}\left(\frac{r}{3} + \frac{br^3}{5R_f^2}\right)$$

7. At the interfaces, the heat flux must be continuous, so at $r = R_f$, $q_r^c = q_r^f$ such that

$$C_1^c = q_r^f R_f^2 = S_{n0}\left(\frac{R_f^3}{3} + \frac{bR_f^3}{5}\right)$$

8. **S**ubstituting the expression for C_1^c yields

$$q_r^c = \frac{R_f^3}{r^2} S_{n0} \left(\frac{1}{3} + \frac{b}{5} \right)$$

9. Now that both components of the heat flux are obtained, the temperatures can be found by using Fourier's Law:

$$-k^f \frac{dT^f}{dr} = S_{n0} \left(\frac{r}{3} + \frac{br^3}{5R_f^2} \right)$$

and

$$-k^c \frac{dT^c}{dr} = \frac{R_f^3}{r^2} S_{n0} \left(\frac{1}{3} + \frac{b}{5} \right)$$

10. Integrating the above equations yields

$$T^f = -\frac{S_{n0}}{k^f} \left(\frac{r^2}{6} + \frac{br^4}{20R_f^2} \right) + C_2^f$$

and

$$T^c = \frac{S_{n0}}{k^c} \left(\frac{1}{3} + \frac{b}{5} \right) \frac{R_f^3}{r} + C_2^c$$

11. At the interfaces, the temperature must be continuous, so at $r = R_f$, $T^f = T^c$, and at $r = R_c$, $T^c = T^0$. Therefore, applying these boundary conditions and solving for the constants yields

$$T^f = \frac{S_{n0} R_f^2}{6k^f} \left[\left[1 - \left(\frac{r}{R_f} \right)^2 \right] + \frac{3}{10} b \left[1 - \left(\frac{r}{R_f} \right)^4 \right] \right] + \frac{S_{n0} R_f^2}{3k^c} \left(1 + \frac{3b}{5} \right) \left(1 - \frac{R_f}{R_c} \right) + T_0$$

and

$$T^c = \frac{S_{n0} R_f^2}{3k^c} \left(1 + \frac{3b}{5} \right) \left(\frac{R_f}{r} - \frac{R_f}{R_c} \right) + T_0$$

8.7 Thermal Resistance

8.7.1 Rectangular R_{th}

- It turns out that many physical phenomena can be described by Flow Rate = Driving Force/Resistance
- Consider heat flowing in the x direction through a rectangular object with length B and area A. If the temperature change is linear,

$$q_x = k \frac{\Delta T}{B} \therefore Q = kA \frac{T_1 - T_2}{B}$$

- Since the change in temperature is the driving force, and Q is the flow rate, the thermal resistance in a rectangular system can be defined as

$$R_{th} = \frac{B}{kA}$$

8.7.2 Defining a General Q Using R_{th}

- Using the above expression for R_{th}, Q can be rewritten as

$$Q = \frac{T_1 - T_2}{R_{th}} = \frac{\Delta T}{R_{th}}$$

  - The above expression holds true for systems in any coordinate system as long as R_{th} is appropriately redefined
  - Here, ΔT is not final minus initial. It is always a positive quantity, and it it is frequently $T_{hot} - T_{cold}$

8.7.3 Cylindrical Shell R_{th}

Problem: Consider a cylindrical tube (with a hollowed out center) with inner radius R_1 and outer radius R_2. The temperature of the innermost surface is T_1 and outermost surface is T_2. Assume heat flows radially and the cylinder has length L. Derive R_{th}.

Solution:

1. Using cylindrical coordinates, the microscopic energy balance can be rewritten as

$$\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right) = 0$$

2. Integrating once yields

$$\frac{dT}{dr} = \frac{C_1}{r}$$

3. Integrating again yields

$$T = C_1 \ln r + C_2$$

4. The temperatures were defined at the interfaces, so at $r = R_1$, $T = T_1$, and at $r = R_2$, $T = T_2$ such that

$$T_1 = C_1 \ln R_1 + C_2$$

and

$$T_2 = C_1 \ln R_2 + C_2$$

5. Solving for C_1 yields

$$C_1 = -\frac{T_1 - T_2}{\ln(R_2/R_1)}$$

   (a) Note that C_2 is not needed (but can easily be solved for) since we are looking for heat flux, and that simply requires C_1 in Step 2 to be found

6. Substituting C_1 in Step 2 and using Fourier's Law yields

$$q_r = \frac{k(T_1 - T_2)}{r\ln(R_2/R_1)}$$

7. The heat flow can then be found as

$$Q = q_r A_r = \frac{2\pi Lk(T_1 - T_2)}{\ln(R_2/R_1)}$$

8. Since the temperature is the driving force, the thermal resistance can be defined as

$$R_{th} = \frac{\ln(R_2/R_1)}{2\pi kL}$$

8.7.4 Spherical Shell R_{th}

- For radial conduction in a spherical system, a similar procedure can be used to find that

$$R_{th} = \frac{R_2 - R_1}{4\pi k R_1 R_2}$$

8.8 Heat Conduction through Composite Walls

8.8.1 Series and Parallel Resistances

- The total resistance of resistances in series is

$$R_{tot} = \sum_i R_i$$

- The total resistance of resistances in parallel is

$$R_{tot} = \left(\sum_i R_i^{-1} \right)^{-1}$$

8.8.2 Series Rectangular Composite

Problem: Consider a rectangular wall composed of three distinct materials. The left third ($x = x_0$ to $x = x_1$) is some arbitrary material 1, the middle ($x = x_1$ to $x = x_2$) is some arbitrary material 2, and the right third ($x = x_2$ to $x = x_3$) is some arbitrary material 3. Each has a unique k value. Find the effective thermal conductivity if $b_1 = x_1 - x_0$, $b_2 = x_2 - x_1$, and $b_3 = x_3 - x_2$.

1. The heat flux must be continuous. For instance, at $x = x_0$, $q_x = q_0$. At the x_1 interface, $q_0 = q_1$. At the x_2 interface, $q_1 = q_2$. At the x_3 interface, $q_2 = q_3$. Therefore, $q_x = q_1 = q_2 = q_3 = q_0$. The heat flux is a constant at each interface.

2. Since this is true, at region 1, 2, and 3,

$$q_0 = -k_{01} \frac{dT}{dx}$$
$$q_0 = -k_{12} \frac{dT}{dx}$$
$$q_0 = -k_{23} \frac{dT}{dx}$$

3. Integrating yields at region 1, 2, and 3,

$$T = -\frac{q_0}{k_{01}} x + C_1$$
$$T = -\frac{q_0}{k_{12}} x + C_2$$
$$T = -\frac{q_0}{k_{23}} x + C_3$$

4. At $x = x_0$, $T = T_0$, and at $x = x_1$, $T = T_1$ such that

$$T_0 - T_1 = \frac{q_0}{k_{01}} (x_1 - x_0)$$

5. Repeating this for the other regions yields

$$T_1 - T_2 = \frac{q_0}{k_{12}} (x_2 - x_1)$$
$$T_2 - T_3 = \frac{q_0}{k_{23}} (x_3 - x_2)$$

6. Adding the temperature from each region yields

$$T_0 - T_3 = q_0 \left(\frac{b_1}{k_{01}} + \frac{b_2}{k_{12}} + \frac{b_3}{k_{23}} \right)$$

7. The heat flow can then be written as

$$Q = q_0 A = \frac{T_0 - T_3}{\left(\frac{b_1}{Ak_{01}} + \frac{b_2}{Ak_{12}} + \frac{b_3}{Ak_{23}}\right)} = \frac{\Delta T}{R_{eff}}$$

where

$$R_{eff} = \frac{b_1}{k_{01}A} + \frac{b_2}{k_{12}A} + \frac{b_3}{k_{23}A} = R_1 + R_2 + R_3$$

8. It is clear that the above system can be modeled with a circuit analog where there are three resistances in series!

9. Bringing the area term to the numerator,

$$Q = \frac{A\left(T_0 - T_3\right)}{\frac{b_1}{k_{01}} + \frac{b_2}{k_{12}} + \frac{b_3}{k_{23}}}$$

10. If we introduce k_{eff} as the effective heat transfer coefficient,

$$Q = \frac{k_{eff}A\left(T_0 - T_3\right)}{b_1 + b_2 + b_3}$$

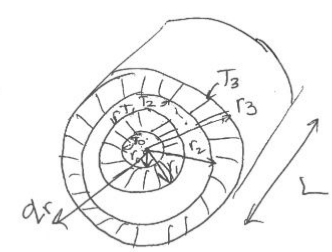
11. Therefore,

$$k_{eff} = \left[\frac{1}{b_1 + b_2 + b_3}\left(\frac{b_1}{k_{01}} + \frac{b_2}{k_{12}} + \frac{b_3}{k_{23}}\right)\right]^{-1}$$

- Note that the previous procedure was mostly a formalism. If you recognize that the composite can be modeled as series resistances, one can jump directly to that step using the previously defined thermal resistances for that coordinate system

8.8.3 Cylindrical Composite

Problem: There are three cylindrical shells surrounding one another. The inner temperature of the first surface from the center is T_1 and is at r_1. The inner temperature of the second surface (outer temperature of the first surface) is T_2 and is at r_2. The inner temperature of the third surface (outer temperature of the second surface) is T_3 and is at r_3. The cylinder has a length L. The heat flux is solely radial. Model the effective thermal resistance. A rough sketch is shown below to help visualize the scenario:

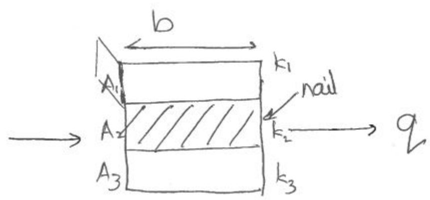


Solution: Using the definition of R_{th} from earlier in cylindrical coordinates and realizing that this is a system in series,

$$R_{eff} = R_1 + R_2 + R_3 = \frac{\ln\left(r_1/r_0\right)}{2\pi k_{01}L} + \frac{\ln\left(r_2/r_1\right)}{2\pi k_{12}L} + \frac{\ln\left(r_3/r_2\right)}{2\pi k_{23}L}$$

8.8.4 Parallel Rectangular Composite

Problem: Consider the following system. Find R_{eff} in terms of k_{eff}.



Solution:

- Since there are now three components in parallel,

$$R_{eff} = \left(R_1^{-1} + R_2^{-1} + R_3^{-1}\right)^{-1} = b\left(\frac{1}{k_{01}A_1} + \frac{1}{k_{12}A_2} + \frac{1}{k_{23}A_3}\right)$$

- Introducing a k_{eff} yields

$$R_{eff} = \frac{b}{k_{eff}\left(A_1 + A_2 + A_3\right)}$$

- To figure out what k_{eff} is equal to, substitute back in for R_{eff}:

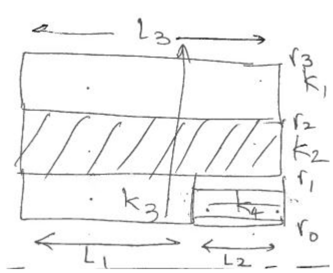
$$\frac{k_{eff}\left(A_1 + A_2 + A_3\right)}{b} = \frac{k_{01}A_1}{b} + \frac{k_2 A_2}{b} + \frac{k_3 A_3}{b}$$

- Therefore,

$$k_{eff} = \frac{\left(k_{01}A_1 + k_{12}A_2 + k_{23}A_3\right)}{A_1 + A_2 + A_3}$$

8.8.5 Series and Parallel Cylindrical Composite

Problem: Consider the following system. Find R_{eff} in terms of k_{eff}. The system below is a rectangular cross-section of a cylinder. Therefore, r_0 is the innermost radius, and r_3 is the outermost radius.



Solution:

- Elements 1 and 2 are in series. Elements 3 and 4 are in parallel. The equivalent of elements 3 and 4 are in series with elements 1 and 2. Therefore,

$$R_{eq-34} = \left(R_3^{-1} + R_4^{-1}\right)^{-1}$$

and

$$R_{eff} = R_1 + R_2 + R_{eq-34}$$

- Using the cylindrical definitions defined earlier,

$$R_{eff} = \frac{\ln(r_1/r_0)}{2\pi k_1 L_3} + \frac{\ln(r_2/r_1)}{2\pi k_2 L_3} + \frac{\ln(r_3/r_2)}{2\pi k_{eq-34} L_3}$$

- Introducing k_{eff} yields

$$R_{eff} = \frac{\ln(r_3/r_0)}{2\pi k_{eff} L_3}$$

- Using the general definition of $Q = \dfrac{\Delta T}{R_{eff}}$,

$$Q = \frac{\Delta T}{\left(\dfrac{\ln(r_3/r_0)}{2\pi k_{eff} L_3}\right)}$$

- To find out what k_{eff} is in this equation, substitute back in for R_{eff}

$$\frac{\ln(r_3/r_0)}{2\pi k_{eff} L_3} = \frac{\ln(r_3/r_2)}{2\pi k_1 L_3} + \frac{\ln(r_2/r_1)}{2\pi k_2 L_3} + \frac{\ln(r_1/r_0)}{2\pi k_{eq-34} L_3}$$

- This simplifies to

$$k_{eff} = \frac{\ln(r_3/r_0)}{\dfrac{\ln(r_3/r_2)}{k_1} + \dfrac{\ln(r_2/r_1)}{k_2} + \dfrac{\ln(r_1/r_0)}{k_{eq-34}}}$$

- However, we have not yet defined k_{eq-34} yet. To find k_{eq-34}, substitute back in for R_{eq-34}. As such,

$$\frac{2\pi k_{eq-34} L_3}{\ln(r_1/r_0)} = \frac{2\pi k_3 L_1}{\ln(r_1/r_0)} + \frac{2\pi k_4 L_2}{\ln(r_1/r_0)}$$

$$k_{eq-34} = \frac{k_3 L_1 + k_4 L_2}{L_1 + L_2} = \frac{k_3 L_1 + k_4 L_2}{L_3}$$

8.9 Newton's Law of Cooling

8.9.1 Definitions

- Newton's law of cooling states the following where T_s is solid surface temperature and T_∞ is the bulk liquid temperature

$$q = h(T_s - T_\infty)$$

  - The constant h is the heat transfer coefficient

- The biot number (dimensionless) is defined as

$$\text{Bi} = \frac{bh}{k} = \frac{\text{heat transfer by fluid}}{\text{heat transfer by solid}}$$

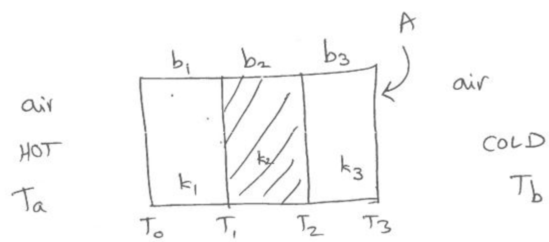
8.9.2 Liquid Bound on One Side

Problem: Consider a bulk liquid at temperature T_∞ bounded on one side by a solid wall at temperature T_s. Derive R_{th}.

1. We know that $Q = \dfrac{\Delta T}{R_{th}}$
2. Using Newton's law of cooling $Q = qA = h(T_s - T_\infty)A$
3. Therefore, $R_{th} = \dfrac{1}{Ah}$ to make $Q = \dfrac{\Delta T}{R_{th}}$

8.9.3 Solid Bound by Two Different Temperature Fluids - Rectangular

Problem: Solve the Series Rectangular Composite problem except for now there is a fluid bounding both sides of the rectangular composite (depicted below):



1. Using the same procedure as in the Series Rectangular Composite question, the following is true

$$T_0 - T_1 = \frac{q_0}{k_{01}} b_1$$

$$T_1 - T_2 = \frac{q_0}{k_{12}} b_2$$

$$T_2 - T_3 = \frac{q_0}{k_{23}} b_3$$

2. Now, we must consider the heat transfer at the solid-liquid interfaces as well. As such,

$$q_0 = h_a \left(T_a - T_0\right)$$

$$q_0 = h_b \left(T_3 - T_b\right)$$

3. The sum of these equations yields

$$T_a - T_b = q_0 \left(\frac{b_1}{k_{01}} + \frac{b_2}{k_{12}} + \frac{b_3}{k_{23}} + \frac{1}{h_a} + \frac{1}{h_b}\right)$$

4. Therefore,

$$Q = q_0 A = \frac{T_a - T_b}{\frac{1}{A}\left(\frac{b_1}{k_{01}} + \frac{b_2}{k_{12}} + \frac{b_3}{k_{23}} + \frac{1}{h_a} + \frac{1}{h_b}\right)}$$

5. Since $Q = \frac{\Delta T}{R_{eff}}$, for the solid region, $R_{th} = \frac{b}{kA}$ as before, and the fluid region is $\frac{1}{hA}$

8.9.4 General Equation

- To generalize the previous examples, a solid rectangular system bounded by fluid can be described by

$$Q = \frac{T_{hot} - T_{cold}}{\frac{1}{A}\left(\frac{1}{h_0} + \sum_{j=1}^{n} \frac{x_j - x_{j-1}}{k_{j-1,j}} + \frac{1}{h_n}\right)}$$

- A solid cylindrical system bounded by fluid can be described by

$$Q = \frac{T_{hot} - T_{cold}}{\frac{1}{2\pi L}\left(\frac{1}{r_0 h_0} + \sum_{j=1}^{n} \frac{\ln\left(r_j / r_{j-1}\right)}{k_{j-1,j}} + \frac{1}{r_n h_n}\right)}$$

- If there is no surrounding fluid to be considered, the terms with h can be dropped

9 The Equations of Change for Nonisothermal Systems

9.1 The Energy Equation

- The general form of the energy equation states that

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho\hat{U}\right) = -\nabla\left[\left(\frac{1}{2}\rho v^2 + \rho\hat{U}\right)\overrightarrow{v}\right] - \nabla\cdot\overrightarrow{q} - \nabla\cdot P\overrightarrow{v} - \nabla\left(\boldsymbol{\tau}\colon\overrightarrow{v}\right) + \rho\left(\overrightarrow{v}\cdot\overrightarrow{g}\right)$$

- The equation of change of temperature states that

$$\rho\hat{C}_p\left(\frac{\partial T}{\partial t} + \overrightarrow{v}\cdot\nabla T\right) = -\nabla\cdot\overrightarrow{q} - \left(\frac{\partial \ln\rho}{\partial \ln T}\right)_p\left(\frac{\partial T}{\partial t} + \overrightarrow{v}\cdot\nabla T\right) - \boldsymbol{\tau} : \nabla\overrightarrow{v}$$

- The following relationships also hold

$$\hat{H} = \hat{U} + \frac{P}{\rho}$$

$$d\hat{H} = \hat{C}_p dT$$

10 Appendix

10.1 Newton's Law of Viscosity

10.1.1 Cartesian

$$\tau_{xy} = \tau_{yx} = -\mu\left[\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right]$$

$$\tau_{yz} = \tau_{zy} = -\mu\left[\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right]$$

$$\tau_{zx} = \tau_{xz} = -\mu\left[\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right]$$

10.1.2 Cylindrical

$$\tau_{r\theta} = \tau_{\theta r} = -\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

$$\tau_{\theta z} = \tau_{z\theta} = -\mu\left[\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right]$$

$$\tau_{zr} = \tau_{rz} = -\mu\left[\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right]$$

10.1.3 Spherical

$$\tau_{r\theta} = \tau_{\theta r} = -\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} = -\mu\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}\right]$$

$$\tau_{\phi r} = \tau_{r\phi} = -\mu\left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right]$$

10.2 Gradient

$$\nabla f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}\,\text{(Cartesian)}$$

$$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{\partial f}{\partial z}\hat{z}\,\text{(Cylindrical)}$$

$$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}\,\text{(Spherical)}$$

10.3 Divergence

$$\nabla \cdot \overrightarrow{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\,\text{(Cartesian)}$$

$$\nabla \cdot \overrightarrow{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}\,\text{(Cylindrical)}$$

$$\nabla \cdot \overrightarrow{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(v_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}\,\text{(Spherical)}$$

10.4 Curl

$$\nabla \times \overrightarrow{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{y} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{z}\,\text{(Cartesian)}$$

$$\nabla \times \overrightarrow{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)\hat{r} + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial (r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta}\right)\hat{z}\,\text{(Cylindrical)}$$

$$\nabla \times \overrightarrow{v} = \frac{1}{r\sin\theta}\left(\frac{\partial (v_\phi \sin\theta)}{\partial \theta} - \frac{\partial v_\theta}{\partial \phi}\right)\hat{r} + \left(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{1}{r}\frac{\partial (r v_\theta)}{\partial r}\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial (r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta}\right)\hat{\phi}\,\text{(Spherical)}$$

10.5 Laplacian

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\,\text{(Cartesian)}$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}\,\text{(Cylindrical)}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}\,\text{(Spherical)}$$

10.6 Continuity Equation

$$[\partial\rho/\partial t + (\nabla \cdot \rho \mathbf{v}) = 0]$$

Cartesian coordinates (x, y, z):

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

Cylindrical coordinates (r, θ, z):

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

Spherical coordinates (r, θ, ϕ):

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 v_r) + \frac{1}{r \sin \theta}\frac{\partial}{\partial \theta}(\rho v_\theta \sin \theta) + \frac{1}{r \sin \theta}\frac{\partial}{\partial \phi}(\rho v_\phi) = 0$$

Note that at constant ρ, the above equations simply become $\nabla \cdot \overrightarrow{v} = 0$

10.7 Navier-Stokes Equation

Cartesian coordinates (x, y, z):

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right] + \rho g_x$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right] + \rho g_y$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$$

Cylindrical coordinates (r, θ, z):

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right] + \rho g_\theta$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$$

Spherical coordinates (r, θ, ϕ):

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right) = -\frac{\partial p}{\partial r}$$
$$+ \mu\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_r}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial \phi^2}\right] + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta - v_\phi^2\cot\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta}$$
$$+ \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(v_\theta\sin\theta)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial \phi^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right] + \rho g_\theta$$

$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r + v_\theta v_\phi\cot\theta}{r}\right) = -\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}$$
$$+ \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(v_\phi\sin\theta)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi} + \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right] + \rho g_\phi$$

10.8 Stream Functions and Velocity Potentials

10.8.1 Velocity Components

$$v_x = -\frac{\partial \psi}{\partial y} = -\frac{\partial \phi}{\partial x},\ v_y = \frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y}\ \text{(Cartesian)}$$

$$v_r = -\frac{1}{r}\frac{\partial \psi}{\partial \theta} = -\frac{\partial \phi}{\partial r},\ v_\theta = \frac{\partial \psi}{\partial r} = -\frac{1}{r}\frac{\partial \phi}{\partial \theta}\ \text{(Cylindrical with } v_z = 0)$$

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial z} = -\frac{\partial \phi}{\partial r},\ v_z = -\frac{1}{r}\frac{\partial \psi}{\partial r} = -\frac{\partial \phi}{\partial z}\ \text{(Cylindrical with } v_\theta = 0)$$

$$v_r = -\frac{1}{r^2\sin\theta}\frac{\partial \psi}{\partial \theta} = -\frac{\partial \phi}{\partial r},\ v_\theta = \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r} = -\frac{1}{r}\frac{\partial \phi}{\partial \theta}\ \text{(Spherical)}$$

10.8.2 Differential Equations

1. Planar Flow

   (a) For Cartesian with $v_z = 0$ and no z-dependence:

$$\frac{\partial}{\partial t}\left(\nabla^2\psi\right) + \frac{\partial\psi}{\partial y}\frac{\partial\left(\nabla^2\psi\right)}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\left(\nabla^2\psi\right)}{\partial y} = \nu\nabla^4\psi$$

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\nabla^4\psi \equiv \left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\partial y^2} + \frac{\partial^4}{\partial y^4}\right)\psi$$

   (b) For cylindrical coordinate with $v_z = 0$ and no z-dependence:

$$\frac{\partial}{\partial t}\left(\nabla^2\psi\right) + \frac{1}{r}\left[\frac{\partial\psi}{\partial r}\frac{\partial\left(\nabla^2\psi\right)}{\partial\theta} - \frac{\partial\psi}{\partial\theta}\frac{\partial\left(\nabla^2\psi\right)}{\partial r}\right] = \nu\nabla^4\psi$$

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

2. Axisymmetrical

   (a) For cylindrical with $v_z = 0$ and no z-dependence:

$$\frac{\partial}{\partial t}\left(E^2\psi\right) - \frac{1}{r}\left[\frac{\partial\psi}{\partial r}\frac{\partial\left(E^2\psi\right)}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial\left(E^2\psi\right)}{\partial r}\right] - \frac{2}{r^2}\frac{\partial\psi}{\partial z}E^2\psi = \nu E^2\left(E^2\psi\right)$$

$$E^2 \equiv \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

   (b) For spherical with $v_\phi = 0$ and no ϕ-dependence:

$$\frac{\partial}{\partial t}\left(E^2\psi\right) + \frac{1}{r^2\sin\theta}\left[\frac{\partial\psi}{\partial r}\frac{\partial\left(E^2\psi\right)}{\partial\theta} - \frac{\partial\psi}{\partial\theta}\frac{\partial\left(E^2\psi\right)}{\partial r}\right] - \frac{2E^2\psi}{r^2\sin^2\theta}\left(\frac{\partial\psi}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial\psi}{\partial\theta}\sin\theta\right) = \nu E^2\left(E^2\psi\right)$$

$$E^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)$$